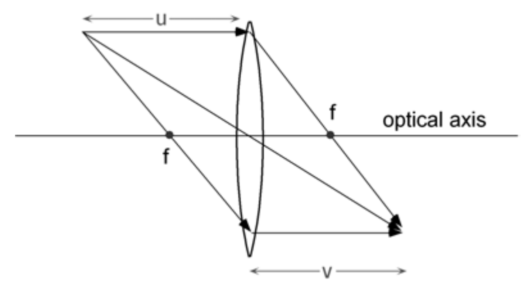
3C43

LASERS
&
MODERN OPTICS

1 Matrix Optics

Analysing optical systems

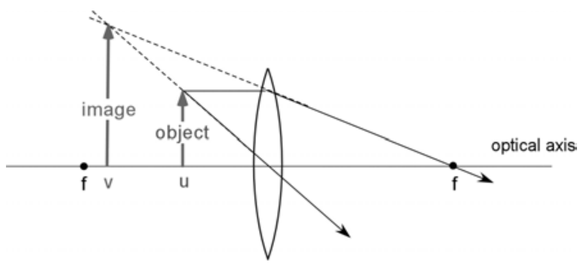
- The simplest optical system is a thin lens.
- We can use a simple geometrical c onstruction for finding location of image formed by a thin lens



- The focal length is the same both sides of the lens
- Thin lens formula: $P = \frac{1}{f} = \frac{1}{u} + \frac{1}{v}$!

Here, u is the object distance to the l.h.s. of the lens, v is the image distance to the r.h.s. of the lens.

• eg. Thin lens as a magnifying glass



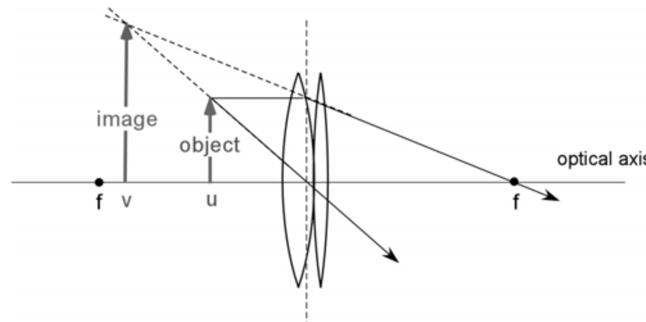
• Thin lens formula:

$$P = \frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

• Magnification,

$$m = \frac{|v|}{u} = \frac{-v}{u} = \frac{f}{f-u} = \frac{1}{1-uP}$$

• Combining thin lenses



• Lens powers add:

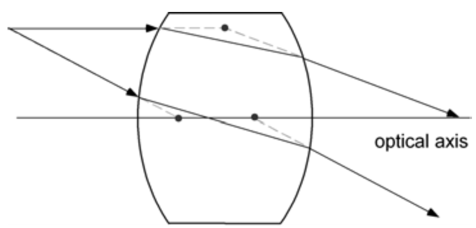
$$P = \frac{1}{f} = P_1 + P_2$$

• Magnification,

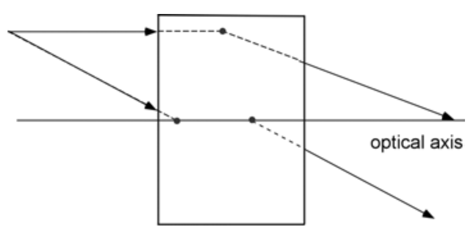
$$m = \frac{|v|}{u} = \frac{-v}{u} = \frac{f}{f-u} = \frac{1}{1-uP}$$

Thick lenses and real optical systems

For thick lenses, the paths of rays outside the lens can still be derived from simple rules, but the rays cannot be considered to be deviated in a single plane:



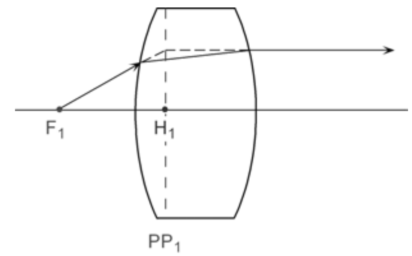
The same is true for entire systems of optical elements (lenses, mirrors)



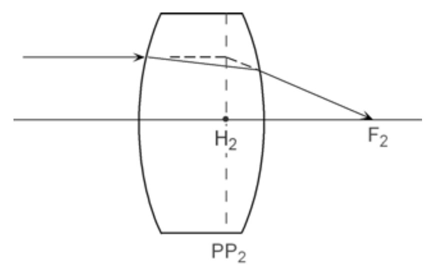
Predicting the effect of the system reduces to that of finding its Cardinal points.

Cardinal points and planes of an optical system

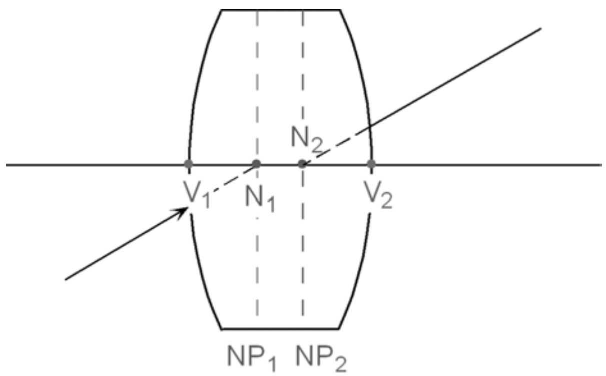
(applies equally well to an arbitrarily complicated optical system)



- PP1 (PP2) = first (second) principal planes
- H1 (H2) = first (second) principal point
- F1 (F2) = first (second) system focal point



- F1 and F2 can be different !



• N1 (N2) = first (second) system nodal point

• NP1 (NP2) = first (second) nodal plane

• V1 (V2) = first (second) system vertices

Ray transfer matrices

How can we characterize the effect of an optical system composed of an arbitrary number of refracting and reflecting elements?

• Paraxial approximation: all rays travel at small angles to the optical axis

• Only meridional rays (propagation in a plane through the optic axis), no skew rays.

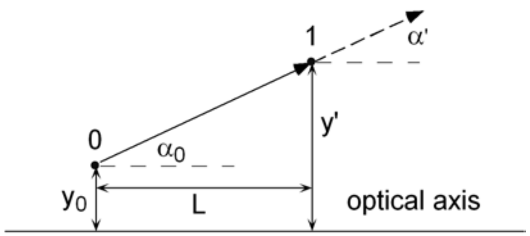
  • changes in height, y, and angle, α, of rays caused by individual elements can be described by linear equations:

$$y' = Ay + B\alpha\ ,\ \ \alpha' = Cy + D\alpha$$

  • individual ray transformations can be represented by a 2x2 matrix

  • the ray transformation caused by a complicated system of optical elements can be found by matrix multiplication.

• Translation matrix



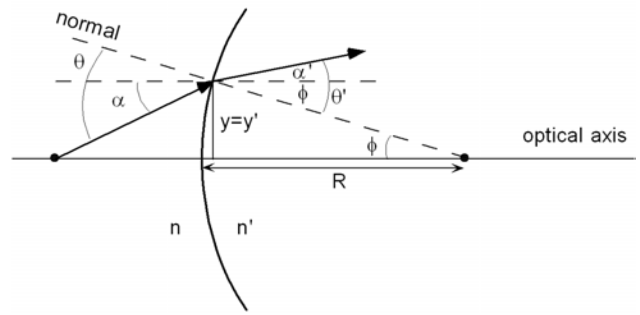
- $\alpha' = \alpha$
- $y' = y + L \tan\alpha$

Paraxial approximation : $\tan\alpha \approx \alpha$

$$\Rightarrow \begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix}$$

- $\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$ is the ray transfer matrix for translation

• Matrix for refraction at a spherical surface



- $\alpha' = \theta' - \phi = \theta' - \dfrac{y}{R}$, $\alpha = \theta - \phi = \theta - \dfrac{y}{R}$
- $y' = y$

• using Snell's law in the paraxial approximation:

$$n\,\theta = n'\,\theta'$$

$$\Rightarrow \quad \alpha' = \frac{1}{R}\left(\frac{n}{n'}-1\right) y + \frac{n}{n'}\alpha$$

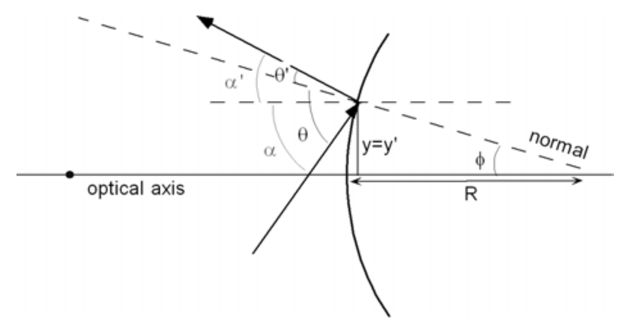
So $$\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'}-1\right) & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'}-1\right) & \frac{n}{n'} \end{bmatrix}$$ is the ray transfer matrix for refraction at a spherical surface (plane surface $R \to \infty$)

❢ R is positive if surface is convex with respect to left-to-right propagating ray, negative if the surface is concave

❢ α is positive for upward-travelling rays, negative for downward

• Matrix for reflection at a spherical surface



❢ R is negative for a concave mirror

❢ α is negative for rays travelling downwards

- $\alpha = \theta - \phi \;\; = \theta - \dfrac{y}{R}$,

- $\alpha' = \theta' + \phi = \theta' + \dfrac{y}{R}$

- Using the rule for reflection $\theta = \theta'$

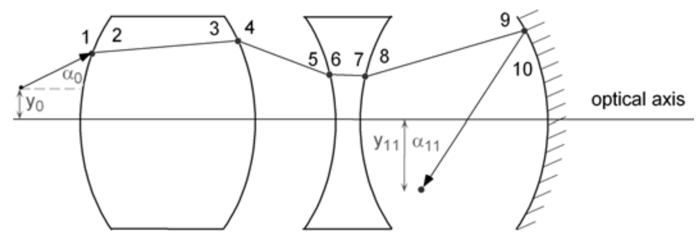
$$\Rightarrow \quad \alpha' = \theta + \frac{y}{R} = \alpha + \frac{2y}{R},$$

$$y' = y$$

or $$\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$ is the ray transfer matrix for reflection at a spherical surface of radius R

System ray transfer matrix



- 0 to 1: translation

$$\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = \underline{\underline{M}}_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

- 1 to 2: refraction at first surface of lens:

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \underline{\underline{M}}_2 \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \underline{\underline{M}}_2 \, \underline{\underline{M}}_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

• generalizing to all 11 steps:

$$\begin{bmatrix} y_{11} \\ \alpha_{11} \end{bmatrix} = \underline{\underline{\mathrm{M}}}_{\,\mathrm{SYSTEM}} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

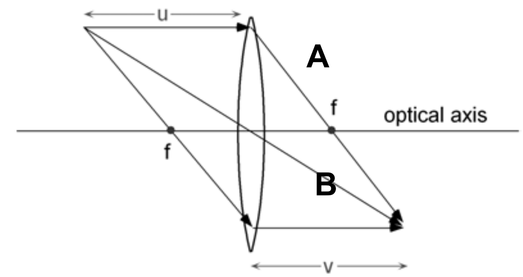
where

$$\underline{\underline{\mathrm{M}}}_{\,\mathrm{SYSTEM}} = \underline{\underline{\mathrm{M}}}_{11}\,\underline{\underline{\mathrm{M}}}_{10}\,\underline{\underline{\mathrm{M}}}_{9} \cdots \underline{\underline{\mathrm{M}}}_{3}\,\underline{\underline{\mathrm{M}}}_{2}\,\underline{\underline{\mathrm{M}}}_{1}$$

• Analysis of entire system reduces to a matrix multiplication.

Ray transfer matrix for a thin lens

• Thin lens (using a ray diagram)



For ray **A** $\begin{pmatrix} y \\ -y/f \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix}$

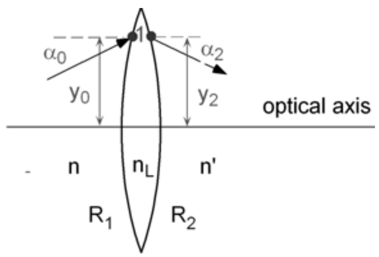
$$\Rightarrow A = 1, \quad C = -1/f$$

For ray **B** $\begin{pmatrix} 0 \\ -\alpha \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 \\ -\alpha \end{pmatrix}$

$$\Rightarrow B = 0, \quad D = 1$$

Thus the transfer matrix is $\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$

• Thin lens (in terms of the elemental refractions



$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \underline{\underline{M}}_2 \, \underline{\underline{M}}_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

where $\underline{\underline{M}}_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n}{n_L} - 1\right) & \frac{n}{n_L} \end{bmatrix}$

(refraction at surface radius R_1)

In this example, R_2 is negative !

$$\underline{\underline{M}}_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n'} - 1\right) & \frac{n_L}{n'} \end{bmatrix}$$

(refraction at surface radius R_2)

So, for the usual case of n = n'

$$\underline{\underline{M}}_{\text{THIN LENS}} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n} - 1\right) & \frac{n_L}{n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n}{n_L} - 1\right) & \frac{n}{n_L} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

where $\frac{1}{f} = \frac{n_L - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

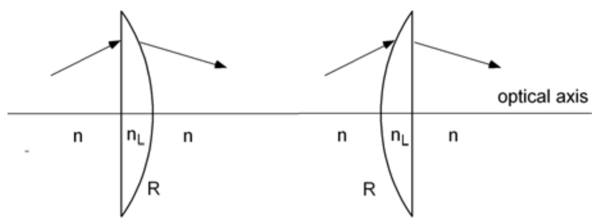
(the lens maker's equation)

Symmetry of ray transfer matrix for a thin lens

Ray transfer matrix for a thin plano-convex lens of

· radius of curvature R,

· refractive index n_L



(a) (b)

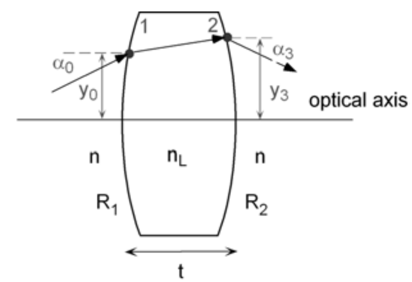
$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

where (a) $\frac{1}{f} = \frac{n_L - n}{n}\left(\frac{1}{\infty} - \frac{1}{-R}\right)$

(b) $\frac{1}{f} = \frac{n_L - n}{n}\left(\frac{1}{R} - \frac{1}{\infty}\right)$

• Thick lens

thickness t, refractive index n_L



$$\underline{\underline{M}}_{\text{THICK LENS}} = \underline{\underline{M}}_3\,\underline{\underline{M}}_2\,\underline{\underline{M}}_1$$

where

$$\underline{\underline{M}}_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n}{n_L} - 1\right) & \frac{n}{n_L} \end{bmatrix}$$

$$\underline{\underline{M}}_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{M}}_3 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n} - 1\right) & \frac{n_L}{n} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\frac{t}{R_1}\left(\frac{n-n_L}{n_L}\right) & t\left(\frac{n}{n_L}\right) \\ \left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left(\frac{n_L-n}{n}\right)-\frac{t(n_L-n)^2}{nn_LR_1R_2} & 1+\frac{t}{R_2}\left(\frac{n_L-n}{n}\right) \end{bmatrix}$$

Messy, but we can extract useful information about the optical system from this matrix – for example, the location of all the cardinal planes - see later.

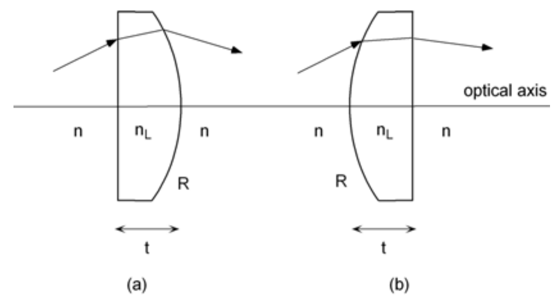
In many cases, the system matrix can be evaluated numerically, which eliminates the cumbersome algebra.

Thick lens:

Find the numerical value of the ray transfer matrix for a thick plano-convex lens of

- radius of curvature R = 10cm,
- refractive index n_L= 1.5,
- thickness t = 1cm

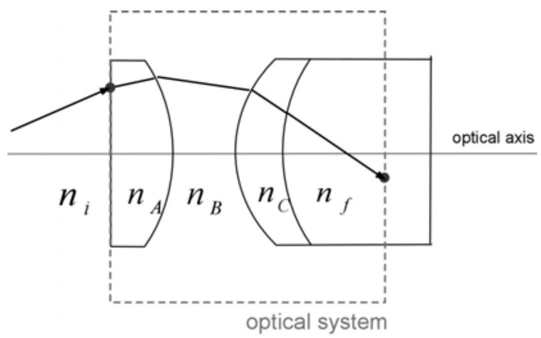
and the determinant of this transfer matrix.



(a) $\underline{\underline{M}} = \begin{bmatrix} 1 & \frac{2}{3}cm \\ -\frac{1}{20cm} & \frac{29}{30} \end{bmatrix}$

(b) $\underline{\underline{M}} = \begin{bmatrix} \frac{29}{30} & \frac{2}{3}cm \\ -\frac{1}{20cm} & 1 \end{bmatrix}$

Determinant of the system transfer matrix



Generally

$$\underline{\underline{M}}_{SYSTEM} = \underline{\underline{M}}_f \ldots \underline{\underline{M}}_3 \underline{\underline{M}}_2 \underline{\underline{M}}_1$$

Thus $\det\left(\underline{\underline{M}}_{SYSTEM}\right) = \prod_{i=1,f} \det\left(\underline{\underline{M}}_i\right)$

= determinant

- Translation through a distance, d

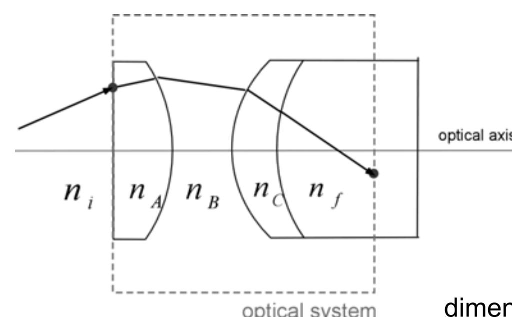
$$\underline{\underline{M}} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \rightarrow \det\left(\underline{\underline{M}}\right) = 1$$

- Refraction at a (generally spherical) interface

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix} \rightarrow \det\left(\underline{\underline{M}}\right) = \frac{n}{n'}$$

- Reflection at a spherical surface

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \rightarrow \det\left(\underline{\underline{M}}\right) = 1$$



dimensionless

So for any system $\det\left(\underline{\underline{M}}\right) = \dfrac{n_i}{n_f}$!

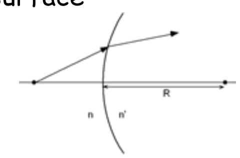
Summary of simple ray transfer matrices

- Translation through a distance, d

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

- Refraction at a spherical surface

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'}-1\right) & \frac{n}{n'} \end{bmatrix}$$



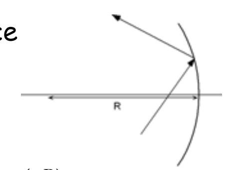
(+R): convex
(-R): concave

- Refraction at a plane surface

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix}$$

- Reflection at a spherical surface

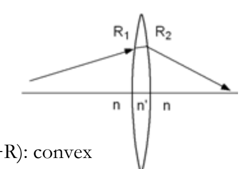
$$\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$



(+R): convex
(-R): concave

- Thin lens matrix

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$



$$f = \frac{n'-n}{n}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

(+R): convex
(-R): concave

Significance of system matrix elements

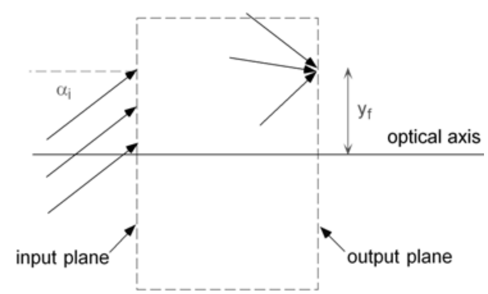
$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

which is equivalent to

$$y_f = A\,y_i + B\alpha_i\,, \quad \alpha_f = C\,y_i + D\alpha_i$$

- Case A=0

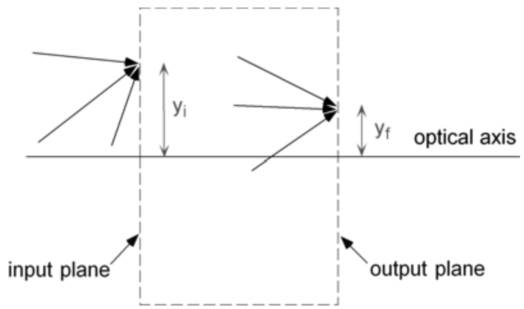
⇨ $y_f = B\alpha_i$



⇨ Output plane = second focal plane

- Case of B=0

⇨ $y_f = A\, y_i$

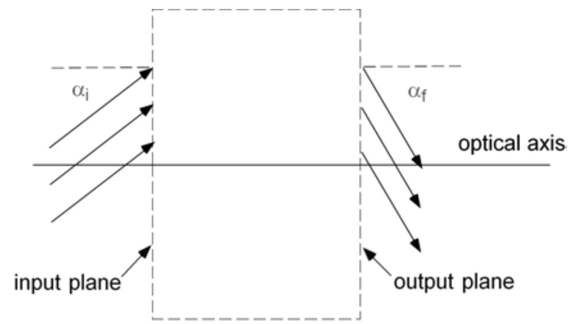


⇨ All rays from the same point y_i in the input plane pass through the same point y_f in the output plane.

Input and output planes are the conjugate (object/image) planes of the system.

- Case of C=0

⇨ $\alpha_f = D\alpha_i$

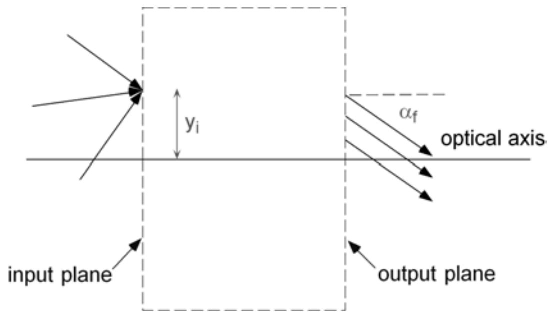


⇨ The system behaves like a telescope with angular magnification

$$m \equiv \frac{\alpha_f}{\alpha_i} = D$$

• Case of D=0

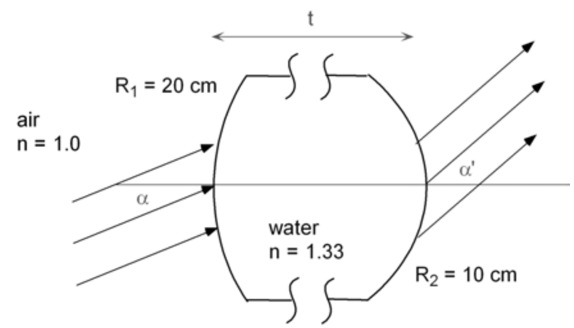
⇨ $\alpha_f = C\, y_i$



⇨ all rays with the same elevation in the input plane leave the output plane at the same angle

⇨ the input plane is the first focal plane of the system

Example



A thin-walled cylinder of length t is filled with water (n=1.33) and has end caps with radii of curvature 20cm and 10 cm.

Show the value of t that allows the cylinder to be used as an astronomical telescope and find the corresponding angular magnification.

Solution

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 \\ \dfrac{n_w - n}{R_2\, n} & \dfrac{n_w}{n} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{n - n_w}{R_1\, n_w} & \dfrac{n}{n_w} \end{bmatrix}$$

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 \\ \dfrac{1/3}{(\text{-10cm}) \times 1} & \dfrac{4/3}{1} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{-1/3}{(\text{20cm}) \times 4/3} & \dfrac{1}{4/3} \end{bmatrix}$$

$$\underline{\underline{M}} = \begin{bmatrix} 1 - t/80cm & 3t/4 \\ -\dfrac{12}{240cm} + \dfrac{t}{2400cm^2} & \dfrac{-3t}{120cm} + 1 \end{bmatrix}$$

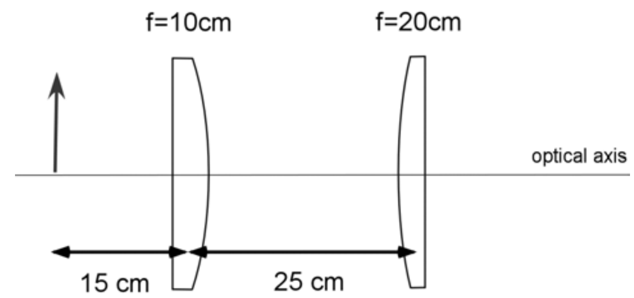
- For system to behave as a telescope C=0

$$\Rightarrow \frac{t}{2400cm^2} = \frac{12}{240cm} \quad \Rightarrow t = 120\ cm$$

then angular magnification

$$\text{M} = \text{D} = \frac{-3 \times 120\ cm}{120cm} + 1 = -2$$

Example



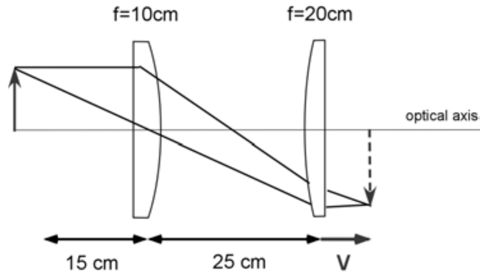
An object is placed 15cm to the left of a thin lens of focal length 10cm, which in turn is 25cm distant from a second thin lens of focal length 20cm.

Find the location and magnification of the image.

Solution

Choose the optical system to lie between then object and image planes.

After constructing the system matrix, we require B=0 for input/output planes to be conjugate planes.



Calculate matrix numerically:

$$\underline{\underline{M}} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{20cm} & 1 \end{bmatrix} \begin{bmatrix} 1 & 25\text{cm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{10cm} & 1 \end{bmatrix} \begin{bmatrix} 1 & 15\text{cm} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{5}{2}cm \\ -\frac{1}{40cm} & -\frac{5}{8} \end{bmatrix}$$

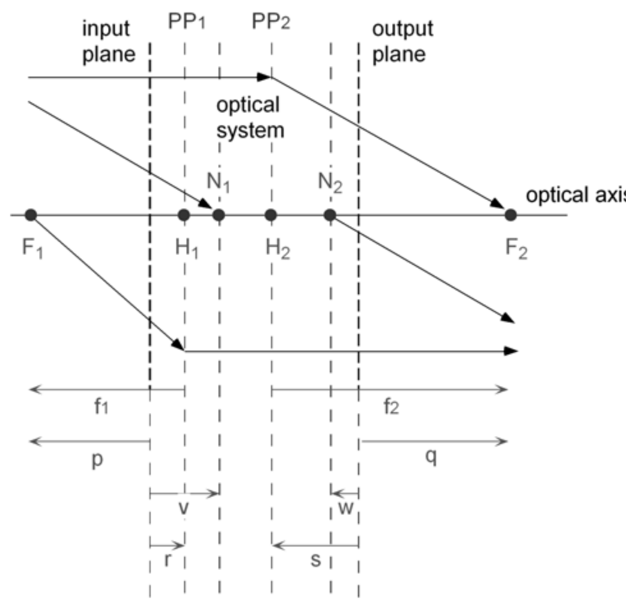
'B' element $= \frac{5}{2}\text{cm} - \frac{5}{8}\text{v} \;\Rightarrow\; v = 4\ cm$

Then magnification = 'A' $= -\frac{3}{2} - \frac{v}{40cm} = -\frac{16}{10}$

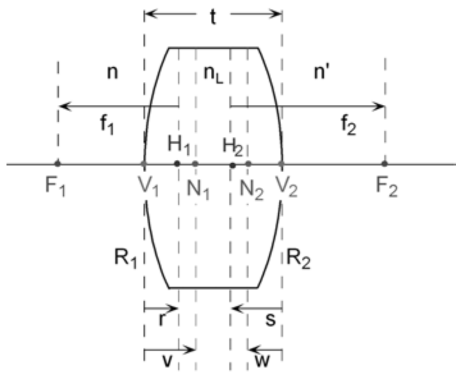
Location of the cardinal points of an optical system

- Derived from the elements of the system transfer matrix

Definitions



Standard coordinates



! Sign convention for distances:

→ = positive

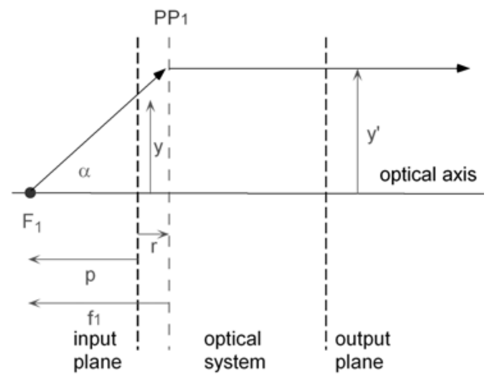
← = negative

! R1, R2 = radii of curvature of lens surfaces

- positive if surface is convex with respect to left-to-right propagating ray,
- negative if surface is concave
- r, s = positions of principal points relative to vertices

- v, w = positions of nodal points relative to vertices
- f1 , f2 = positions of focal points relative to principal points

- first focal plane



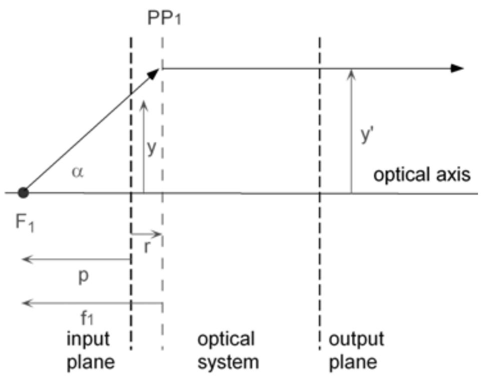
$$y' = A(y) + B(\alpha)$$

- $\alpha' = 0 = C(y) + D(\alpha)$

- $y = -\dfrac{D}{C}\,\alpha$

  ⇨ $\alpha = \dfrac{y}{|p|} = \dfrac{y}{-p}$

- from diagram, $p = \dfrac{-y}{\alpha} = \dfrac{D}{C}$



$$y' = |f_1|\,\alpha = (-f_1)\,\alpha$$

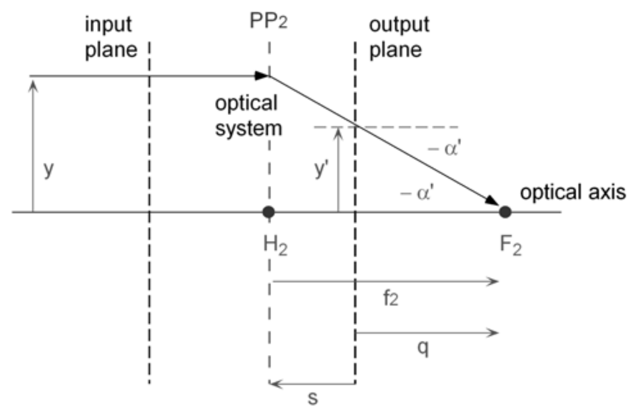
⇨ $$f_1 = \frac{-y'}{\alpha} = -\frac{Ay + B\alpha}{\alpha} = \frac{AD}{C} - B$$

$$= \frac{1}{C} Det(M) = \frac{1}{C}\frac{n}{n'}$$

⇨ $$r = -(f_1 - p) = -\frac{n}{n'}\frac{1}{C} + \frac{D}{C}$$

$$= \frac{1}{C}\left(D - \frac{n}{n'}\right)$$

- second focal plane

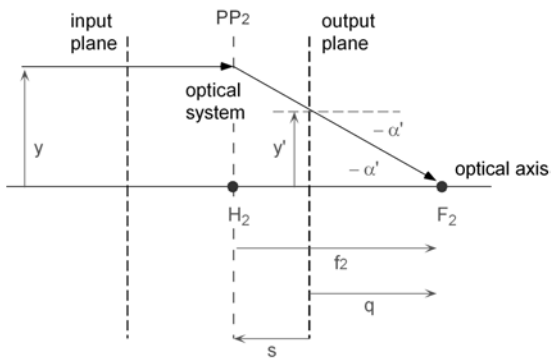


- $y' = A(y) + B(0)$
- $\alpha' = C(y) + D(0)$

⇨ $$y' = \frac{A}{C}\,\alpha'$$

- from diagram, $\alpha' = -\dfrac{y'}{|q|} = -\dfrac{y'}{q}$

⇨ $$q = \frac{-y'}{\alpha'} = -\frac{A}{C}$$

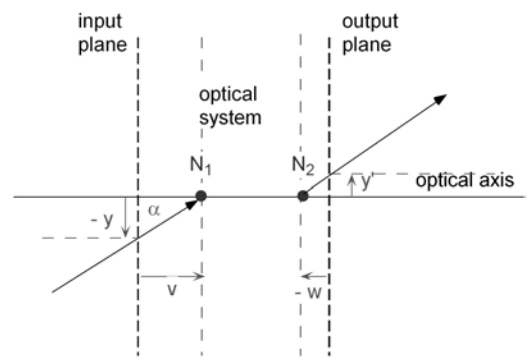


- $f_2 = \dfrac{y}{-\alpha'} = \dfrac{y}{-\left[Cy + D(0)\right]} = -\dfrac{1}{C}$

⇨ $s = -\,(f_2 - q)$

$= -\,\left(\dfrac{-1}{C} - \dfrac{-A}{C}\right) = \dfrac{1-A}{C}$

- nodal planes



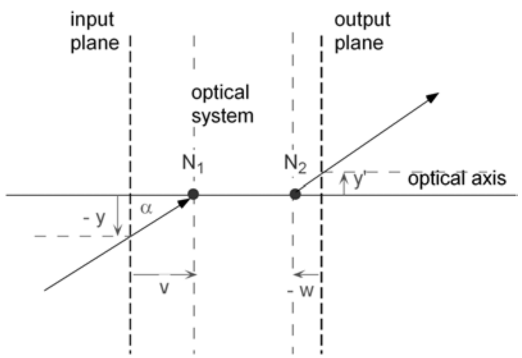
- $y' = A(y) + B(\alpha)$

- $\alpha' = Cy + D\alpha = \alpha$

⇨ $\dfrac{y}{\alpha} = \dfrac{1-D}{C}$

- from diagram, $\dfrac{|y|}{v} = \dfrac{-y}{v} = \alpha$

⇨ $v = \dfrac{-y}{\alpha} = \dfrac{D-1}{C}$



- $y' = Ay + B\alpha$
- $\alpha' = Cy + D\alpha \;\; = \alpha$

  ⇨ $\dfrac{y}{\alpha} = \dfrac{1-D}{C}$

- $\alpha = \alpha'$ implies $\dfrac{y'}{-w} = \dfrac{-y}{v}$

  ⇨ $w = \dfrac{n/n' - A}{C}$

- Locations of cardinal points - summary

$$p = \frac{D}{C}, \quad q = -\frac{A}{C}, \quad r = \frac{D - n/n'}{C}, \quad s = \frac{1-A}{C}, \quad v = \frac{D-1}{C}, \quad w = \frac{n/n' - A}{C}$$

relative to input/output planes

$$f_1 = \frac{n/n'}{C}, \quad f_2 = -\frac{1}{C}$$

relative to principal planes

Ray-tracing

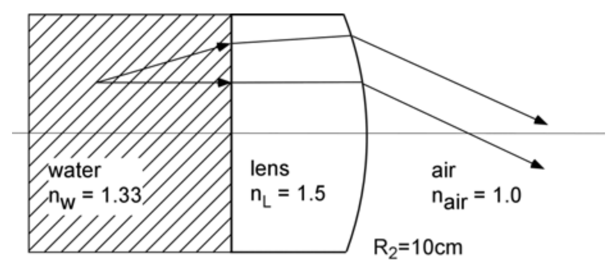
- Computers allow true trajectories of rays to be evaluated numerically, to high accuracy, without recourse to the approximation of paraxial rays.

- Such an analysis is straightforward for meridional rays (which are incident in planes containing the optical axis) as it then reduces to a two-dimensional model: the rays remain in the plane of incidence.

- The formation of an image usually involves rays travelling at small angles to the system optical axis, so that the transfer matrix formalism is useful in practice.

Example

Determine the focal lengths and principal points for a 4cm thick, plano-convex lens with refractive index 1.5 and radius of curvature 10cm when the lens caps the end of a long cylinder filled with water (n=1.33)



Solution

The thick lens transfer matrix is

$$\underline{\underline{M}}_{\text{THICK LENS}} = \underline{\underline{M}}_3\ \underline{\underline{M}}_2\ \underline{\underline{M}}_1$$

$$\underline{\underline{M}}_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n_w}{n_L}-1\right) & \frac{n_w}{n_L} \end{bmatrix}$$

$$\underline{\underline{M}}_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{M}}_3 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n_{air}}-1\right) & \frac{n_L}{n_{air}} \end{bmatrix}$$

Notice that because of the different refractive indices, an analytic solution is very messy!

So, do it numerically

$$\underline{\underline{M}}_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n_w}{n_L}-1\right) & \frac{n_w}{n_L} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 8/9 \end{bmatrix}$$

$$\underline{\underline{M}}_2 = \begin{bmatrix} 1 & 4cm \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{M}}_3 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n_{air}}-1\right) & \frac{n_L}{n_{air}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{10cm}\left(\frac{1}{2}\right) & \frac{3}{2} \end{bmatrix}$$

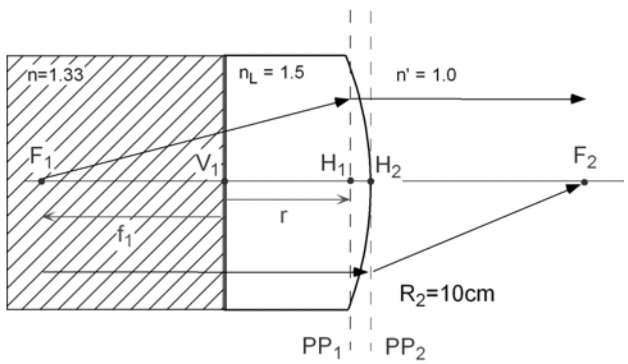
$$\underline{\underline{M}}_{\text{THICK LENS}} = \underline{\underline{M}}_3\ \underline{\underline{M}}_2\ \underline{\underline{M}}_1 = \begin{pmatrix} 1 & \frac{32cm}{9} \\ \frac{-1}{20cm} & \frac{104}{90} \end{pmatrix}$$

Then $\frac{n_{air}}{n_w}\frac{1}{f_1} = -\frac{1}{f_2} = C = -\frac{1}{20cm}$

or $f_1 = -22.5cm, \quad f_2 = 20cm$

Also $r = \dfrac{D - {}^{n}\!/_{n'}}{C} = \dfrac{32}{9} cm$

$$s = \frac{1-A}{C} = 0$$



Example

- Find the system transfer matrix for two thin lenses of focal lengths f_A and f_B separated by a distance L in air.
- Show that the equivalent focal length of the system is

$$\frac{1}{f_{eq}} = \frac{1}{f_A} + \frac{1}{f_B} - \frac{L}{f_A f_B}$$

- A Huygens eyepiece consists of two thin-lenses separated by the average of their focal length.

  Draw a scale diagram showing the locations of the cardinal points for such a system with
  - f_A =8cm
  - f_B = 4cm
- Using the diagram, determine the nature and location of the image of an object placed 6 cm to the left of lens A.

Solution

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_B} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_A} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{L}{f_A} & L \\ \frac{1}{f_B}\left(\frac{L}{f_A}-1\right)-\frac{1}{f_A} & 1-\frac{L}{f_B} \end{bmatrix}$$

$$\frac{1}{f_{eq}} = |\mathrm{C}| = \frac{1}{f_A} + \frac{1}{f_B} - \frac{L}{f_A f_B}$$

For a system with

- f_A =8 cm
- f_B = 4 cm
- L = 6 cm

$$\underline{\underline{M}}_{SYSTEM} = \begin{pmatrix} \frac{1}{4} & 6cm \\ -\frac{3}{16\,cm} & \frac{-1}{2} \end{pmatrix}$$

$$p = \frac{D}{C} = 8/3 \ cm$$

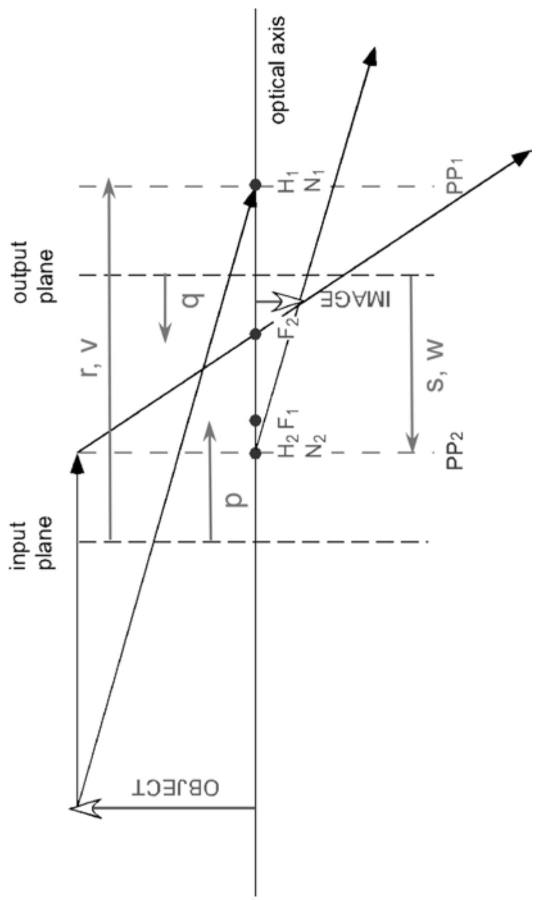
$$q = -\frac{A}{C} = -4/3 \ cm$$

$$r = v = \frac{D\text{-}1}{C} = 8cm$$

$$s = w = \frac{1-A}{C} = -4cm$$

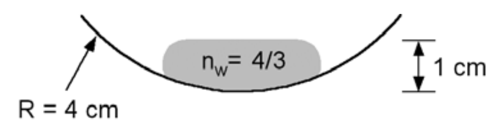
$$f_{equiv} = \frac{1}{C} = 16/3 \ cm$$

The image is inverted and inside the eyepiece.



Example

A water droplet is placed on a thin elastic membrane and is used as a lens to view an object placed far below the membrane. In equilibrium, the membrane has a radius of curvature at its lowest point of 4cm and the water drop is then of thickness 1cm.



1. Show that the optical effect of the membrane layer can be neglected, whatever its refractive index if it can be assumed to be sufficiently thin.

2. Find the ray transfer matrix for the lens and hence the locations of the principal, nodal planes and focal planes.

3. How far above the water surface is the image formed?

Solution

1. If the membrane is thin then we only have to take into account refraction at its two interfaces.

Let n = refractive index of membrane,
n_1 (n_2) = refractive index of medium before(after) membrane

$$\Rightarrow \underline{\underline{M}}_{membrane} = \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n-n_2}{n_2}\right) & \frac{n}{n_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1-n}{n}\right) & \frac{n_1}{n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1-n_2}{n_2}\right) & \frac{n_1}{n_2} \end{pmatrix}$$

Which is the same as the matrix for an interface $n_1 \rightarrow n_2$ of the same radius of curvature.

$$\underline{\underline{M}}_{LENS} = \begin{pmatrix} 1 & 0 \\ \frac{1}{R_2}(n_w-1) & n_w \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{1-n_w}{n_w}\right) & \frac{1}{n_w} \end{pmatrix}$$

translation

refraction at upper surface

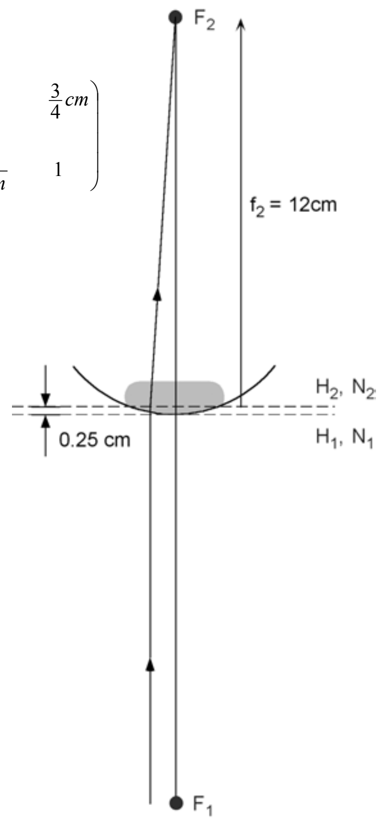
refraction at lower surface

with $t = 1\,cm, \quad R_1 = 4\,cm, \quad R_2 = \infty, \quad n_w = 4/3$

$$\underline{\underline{M}}_{LENS} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 1cm \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{16\,cm} & \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{15}{16} & \frac{3}{4}cm \\ -\frac{1}{12\,cm} & 1 \end{pmatrix}$$

With $\underline{\underline{M}}_{LENS} = \begin{pmatrix} \frac{15}{16} & \frac{3}{4}cm \\ -\frac{1}{12\,cm} & 1 \end{pmatrix}$



$$p = \frac{D}{C} = -12\ cm$$

$$q = -\frac{A}{C} = \frac{45}{4}\ cm$$

$$r = v = \frac{D\text{-}1}{C} = 0$$

$$s = \frac{1-A}{C} = -\frac{3}{4}cm$$

$$f_1 = \frac{1}{C} = -12cm$$

$$f_2 = -\frac{1}{C} = 12cm$$

3. Distance between water surface (output plane) and image (at F_2) is q = 11.25 cm.