3C43

LASERS
&
MODERN OPTICS

3 Gaussian beams

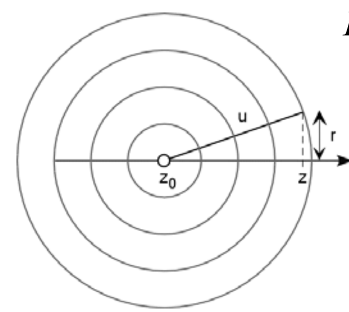
Gaussian beams

- A treatment based on rays cannot explain the spatial characteristics of laser beams.
- A proper description can only be obtained by returning to the wave-equation and finding solutions matching the boundary conditions imposed by a laser cavity.
- (Scalar) wave equation:

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

- Spherical waves emitted by a point source:

$$E \propto \frac{e^{iku}}{u}$$



• Assume the source is at $z=z_0$ and look for a solution valid close to the z-axis

$$u=(z-z_0)\cdot\left(1+\frac{r^2}{(z-z_0)^2}\right)^{1/2}, \qquad r^2\equiv x^2+y^2$$

$$\approx(z-z_0)\cdot\left(1+\frac{r^2}{2(z-z_0)^2}\right)$$

• Substituting into the formula for the spherical wave:

$$E\propto\frac{\exp ik(z-z_0)\cdot\exp\left(\frac{ikr^2}{2(z-z_0)}\right)}{(z-z_0)\cdot\left(1+\frac{r^2}{2(z-z_0)^2}\right)}$$

The term that gives the contribution to the phase resulting from the curvature of the wave-fronts is:

$$\exp\left(\frac{ikr^2}{2R(z)}\right)$$

where $R(z) = z-z_0$ is the radius of curvature of the wave-fronts at z

• Mathematically, this solution is just as valid if we set $z_0=ib$, where b is assumed real (It's still a solution of the wave-equation)

The justification for doing this is that it turns out this is the solution we want.

With $z_0=ib$ we find:

$$E\propto\frac{\exp ikz\cdot\exp\left(\frac{ikr^2}{2(z-ib)}\right)}{(z-ib)\cdot\left(1+\frac{r^2}{2(z-ib)^2}\right)}$$

This is a product of four factors. Examine each in turn.

• $\exp ikz$, just the usual plane-wave phase-factor

- $\exp\left(\dfrac{ikr^2}{2(z-ib)}\right)$, the most important term

(z-ib) is known as the complex radius of curvature (usually represented by q)

$$\exp\left(\frac{ikr^2}{2(z-ib)}\right)=\exp\left(\frac{ikr^2\cdot(z+ib)}{2(z^2+b^2)}\right)$$

$$=\exp\left(\frac{ikzr^2}{2(z^2+b^2)}\right)\exp\left(\frac{-kbr^2}{2(z^2+b^2)}\right)$$

$$= \quad \exp\left(\frac{ikr^2}{R(z)}\right)\cdot\exp\left(\frac{-r^2}{w^2(z)}\right)$$

w(z) characterises the transverse width of the beam

$$w^2(z)=\frac{2(z^2+b^2)}{kb}$$

at z=0 $\quad w^2(0)=w_0^2=\dfrac{2b}{k}=\dfrac{\lambda b}{\pi}$

Hence $\quad w^2(z)=w_0^2\left(1+\dfrac{z^2\lambda^2}{\pi w_0^4}\right)$

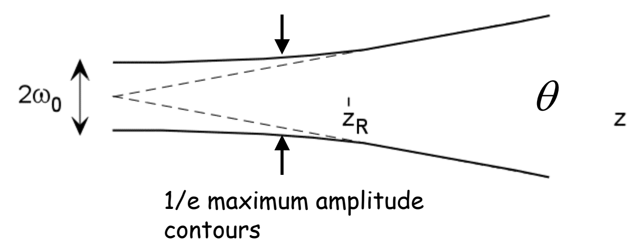
Defining the Rayleigh range $\quad z_R\equiv b=\dfrac{\pi w_0^2}{\lambda}$

we have $\quad w^2(z)=w_0^2\left(1+\dfrac{z^2}{z_R^2}\right)$

For $z \gg z_R$ $\quad w(z)\approx w_0\dfrac{z}{z_R}$

And the divergence of the beam approaches a constant value:

$$\theta=\frac{2w(z)}{z}=\frac{2w_0}{z_R}=\frac{2\lambda}{\pi w_0}$$



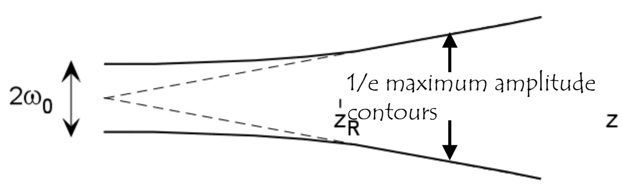
R(z) is the (real) radius of curvature of the wave-fronts at position z:

$$R(z) = z\left(1 + \frac{b^2}{z^2}\right) = z\left(1 + \frac{\pi^2 w_0^4}{\lambda^2 z^2}\right)$$

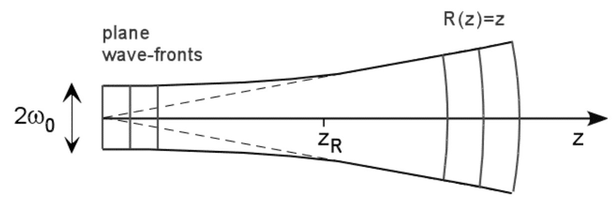
$$= z\left(1 + \frac{z_R^2}{z^2}\right)$$

at z=0 $R(0) = \infty$, a plane wave-front

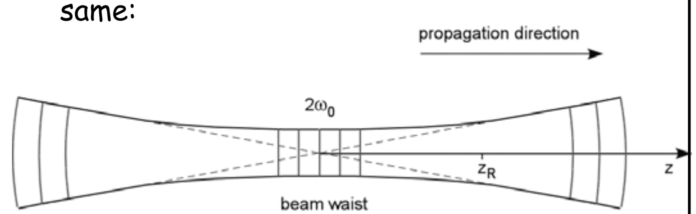
at large z ($z \gg z_R$) $R(z) = z$, as per a spherical wave



So the Gaussian beam propagating to the right of z=0 has the form:



In z < 0, the beam profile is just the same:



Terminology:

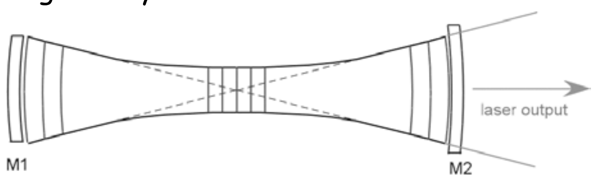
- The beam spot-size is the 1/e half-width (amplitude) at any point z.
- The beam waist, w_0 is the spot-size at z=0.
- The beam has a waist at z=0.

Gaussian beams and laser modes

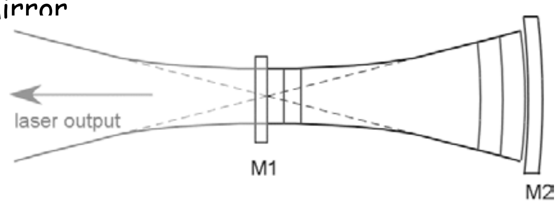
- Which Gaussian beam represents a laser mode?

  The one for which the wave-fronts match the curvatures of the cavity mirrors

e.g. Cavity with two curved mirrors



e.g. Cavity with one plane and one curved mirror



Laser beams have low divergence (usually) because the mirror parameters are chosen to give large z_R.
Not all combinations of mirrors will do!

- Other terms in $E \propto \dfrac{\exp ikz \cdot \exp\left(\dfrac{ikr^2}{2(z-ib)}\right)}{(z-ib)\cdot\left(1+\dfrac{r^2}{2(z-ib)^2}\right)}$
- $E \propto \dfrac{1}{z-ib} = \dfrac{1}{z-iz_R}$
- for large z, just the usual 1/u dependence of the amplitude, as expected for a spherical-wave.

But with a difference:

The contribution of this term to the phase of field is

$$\psi(z) = \frac{\pi}{2} - \arctan\left(\frac{z}{z_R}\right)$$

There is an additional phase shift of π (compared to the plane-wave case), when passing through the beam waist, from $z=-\infty$ to $z=+\infty$, the Guoy phase.

- $E \propto \dfrac{1}{\left(1+\dfrac{r^2}{2(z-ib)^2}\right)}$

for large z, we can neglect b and, for given r, the denominator gets smaller for increasing z.

This just means the intensity of the beam drops off more slowly with distance than does that of a spherical wave (because the beam is more confined).

- How can we locate the position of the beam waist and it size?

We need to find *1/q*, the reciprocal of the complex radius of curvature, since

$$q = z - ib$$

$$\exp\left(\frac{ikr^2}{2q}\right) = \exp\left(\frac{ikr^2\cdot(z+ib)}{2(z^2+b^2)}\right)$$

$$= \exp\left(\frac{ikr^2}{R(z)}\right)\cdot\exp\left(\frac{-r^2}{w^2(z)}\right)$$

Hence $\dfrac{1}{q} = \dfrac{2}{ik}\left[\dfrac{-1}{w^2(z)} + \dfrac{ik}{2R(z)}\right]$

$$\frac{1}{q} = \frac{1}{R(z)} + i\frac{\lambda}{\pi\, w^2(z)} \quad !$$

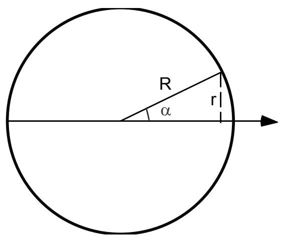
So we can find the waist by setting

$\mathrm{Re}\left\{\dfrac{1}{q}\right\} = 0$ or $\mathrm{Re}\{q\} = 0$

ABCD Law of Gaussian beams

- How is q affected by propagation through an optical system (which could be a laser cavity)?

As before, consider first a spherical wave, with a real radius of curvature, then transform to the complex case.



$r = R \cdot \tan\alpha \approx R\alpha$ (paraxial approximation)

Hence $R \approx \dfrac{r}{\alpha}$

If at the entrance (exit) face of an optical system, r and a have the values r_0, α_0 (r_1, α_1) we find, using the system ray transfer matrix:

$$R_1 = \frac{r_1}{\alpha_1} = \frac{Ar_0 + B\alpha_0}{Cr_0 + D\alpha} = \frac{AR_0 + B}{CR_0 + D}$$

- For Gaussian beams, the role of R(z) for a spherical-wave is taken by q, the complex radius of curvature.

So q is transformed by an optical system according to:

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \quad !$$

where ABCD are the usual elements of the system ray transfer matrix.

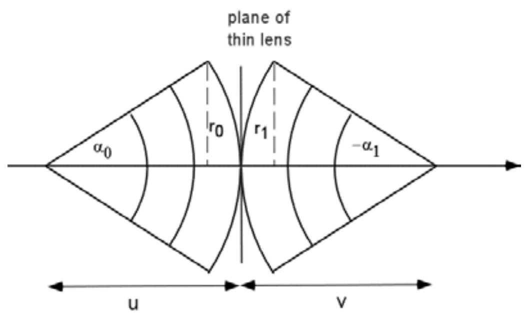
This is known as the
ABCD-law of Gaussian beams.

To find the location and size of a beam waist, it is sometimes more useful to see how $1/q$ transforms:

$$\frac{1}{q_1} = \frac{C + D\dfrac{1}{q_0}}{A + B\dfrac{1}{q_0}}$$

Gaussian beams and lenses

• For the case of a spherical-wave we have, with the usual sign convention for the angles α_0 and α_1 :



$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f} \Rightarrow \frac{1}{R_0}-\frac{1}{R_1}=\frac{1}{f}$$

NB - the radius of curvature of a Gaussian beam before a beam waist is negative

• Ray transfer matrix for a (thin) lens:

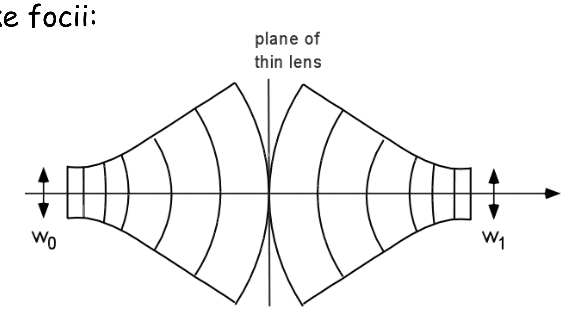
$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

So ABCD-law gives $\frac{1}{q_1}=-\frac{1}{f}+\frac{1}{q_0}$

or $\frac{1}{q_0}-\frac{1}{q_1}=\frac{1}{f}$

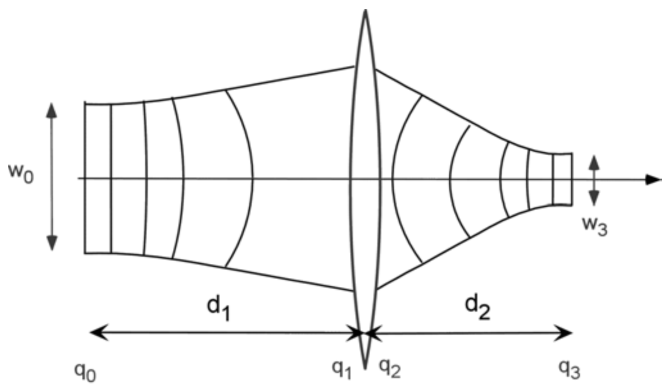
identical to the relation we found for an incident spherical-wave, but with R replaced by *q*.

However, with a Gaussian beam there are waists at the conjugate points, not point-like focii:



Focusing and spot-size

Lens of focal length f placed a distance d_1 from the beam waist. Find the location and size of the waist after the lens.



At first waist $q_0 = -ib_0 = -i\frac{\pi w_0^2}{\lambda}$

L.H.S. of lens $q_1 = d_1 - ib_0$

R.H.S. of lens

$$q_2 = \frac{f\,q_1}{f - q_1} = \frac{f\left(d_1 - ib_0\right)}{f - \left(d_1 - ib_0\right)}$$

Consider the case of $|f - d_1| << |b_0|$

Then $q_2 = \frac{f\left(d_1 - ib_0\right)}{f - \left(d_1 - ib_0\right)} \approx -f - i\frac{f\,d_1}{b_0}$

$$\approx -f - i\frac{f^2}{b_0}$$

and $q_3 = q_2 + d_2 = \left(d_2 - f\right) - i\frac{f^2}{b_0}$

Second waist is located where $\mathrm{Re}\{q\} = 0$

or $d_2 = f$

Then $q_3 = -i\frac{f^2}{b_0} = -i\frac{\pi\,w_3^2}{\lambda}$

or $w_3 = \frac{\lambda\,f}{\pi\,w_0}$ **!**

Gaussian beams & laser cavity modes

The stability criterion for a laser cavity can be found using the ABCD-law.

If $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the ray transfer matrix for a complete round-trip (starting from any point in the cavity), we must have, for a stable mode:

$$q = \frac{Aq+B}{Cq+D}$$

Hence $\frac{1}{q} = \frac{D-A}{2B} \pm \frac{i}{2B} \cdot \sqrt{4-(D+A)^2}$

Since $\Im m\left(\frac{1}{q}\right) = \frac{\lambda}{\pi w^2}$, for a finite spot-size

the condition for a stable mode to exist is

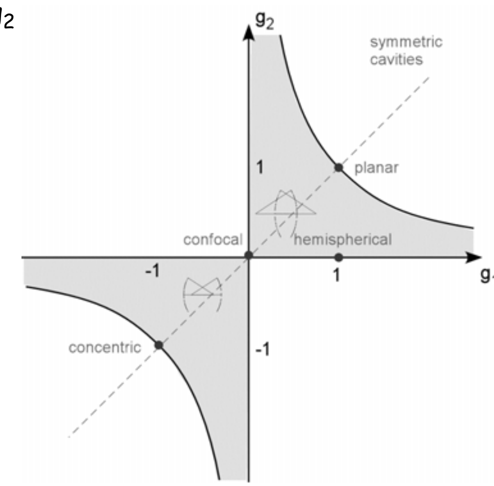
$$|D+A| < 2 \quad !$$

For a cavity formed by mirrors with radii of curvature R_1 and R_2 separated by distance L the condition is equivalent to

$$0 < g_1 g_2 \equiv (1-\frac{L}{R_1})(1-\frac{L}{R_2}) < 1 \quad !$$

Classes of laser cavity

The set of stable cavities can be represented as below on a plot of g_1 versus g_2



Near planar	Confocal
$g \to +1, \;\; w_0, w_1, w_2 \to \infty$	$g = 0, \;\; w_0 = \sqrt{L\lambda/2\pi}$, $w_1, w_2 = w_0\sqrt{2}$
Concentric	**Hemispherical**
$g \to -1, \;\; w_0 \to 0,$ $w_1, w_2 \to \infty$	$g_1 = +1, \;\; g_2 = 0,$ $w_0 = w_1 \to 0, \;\; w_2 \to \infty$

• Symmetric cavities ($g_1 = g_2$)

The symmetry demands that the waist is at the centre of the cavity.

We find for the waist and mirror spot sizes:

$$w_0 = \left(\frac{L\lambda}{2\pi}\right)^{1/2} \cdot \left(\frac{1+g}{1-g}\right)^{1/4}$$

$$w_1 = w_2 = \left(\frac{L\lambda}{\pi}\right)^{1/2} \cdot \left(\frac{1}{1-g^2}\right)^{1/4}$$

For the symmetric confocal cavity g = 0 and so

$$w_1 = w_2 = w_0\sqrt{2}$$

• Typical magnitudes:

$$L \approx 10cm,\ \lambda \approx 1\mu m \Rightarrow \left(\frac{L\lambda}{\pi}\right)^{1/2} \approx 0.5mm$$

Laser beams generally have very low divergence.

Higher-order transverse modes

• More general solutions of the wave-equation can be written

$$E(x,y,z) \propto H_m\left(\frac{x\sqrt{2}}{w(z)}\right) \cdot H_n\left(\frac{y\sqrt{2}}{w(z)}\right) \cdot \exp\frac{ik(x^2+y^2)}{2(z-ib)}$$

where the solution can be identified as a TEM_{mn} transverse mode. w(z) and b are given by the same expressions as before.

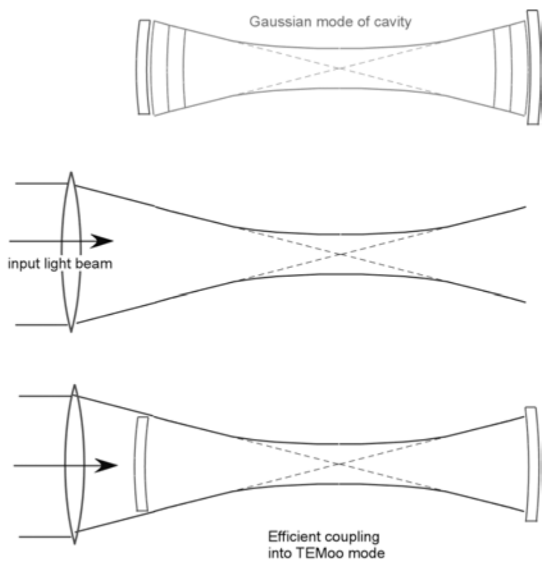
$H_n(x)$ is a Hermite polynomial with n nodes.

For $m=n=0$ we recover the TEM_{00} Gaussian beam.

Mode-matching

- Efficient coupling of radiation in to a single transverse cavity mode, usually the TEM_{00} mode.

- Achieved by a using an external lens to produce a waist of the appropriate size at the appropriate location in the cavity.



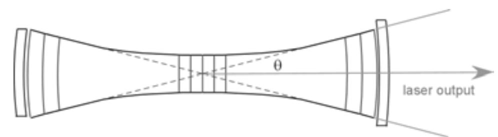
Geometrical properties of laser light

- Directionality (pointing accuracy)

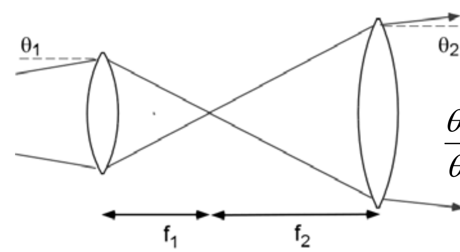
  Fundamental limit to directionality is set by the beam divergence due to diffraction.

  For a TEM_{00} Gaussian mode $\theta_{1/2} \approx \dfrac{\lambda}{\pi w_0}$

  with w_0 the intra-cavity waist size.



The divergence generally increases for higher transverse modes but can be improved by beam collimation using a telescope:



$$\frac{\theta_2}{\theta_1} \approx \frac{f_1}{f_2}$$

- Focusability

If a collimated beam of radius w_0 is incident on a lens of focal length *f* we have for resulting waist size:

$$w \approx \frac{f\,\lambda}{\pi w_0}$$

With $w_0 \approx D$, the lens diameter, we get:

$$w \approx \frac{f\,\lambda}{\pi D} = \frac{\lambda}{\pi} F$$

where *F* is the numerical aperture of the lens ($F \equiv \frac{f}{D}$)

Typically, $F \geq 1$ so $w_{\min} \approx \lambda$

- Brightness

Brightness is defined as the power emitted per unit area of a source per unit solid angle.

e.g. brightness of the solar surface:

Assume the radiation leaving the solar surface to be blackbody radiation in equilibrium at 6000K, having a peak wavelength around 600 nm and a spectral linewidth of 100 nm.

Starting from

$$\rho(\nu)\,d\nu = \frac{8\pi\,h\,\nu^3}{c^3}\,\frac{d\nu}{\exp\left(\frac{h\,\nu}{k_B T}\right) - 1}$$

the spectral energy density of black-body radiation.

the irradiance (power emitted per unit area) at the suns surface is

$$I \approx \tilde{\rho}(\lambda) c\,\Delta\lambda = \frac{8\pi\,hc}{\lambda_0^3}\frac{1}{\exp\left(\dfrac{hc}{k_B T \lambda_0}\right)-1}\frac{c}{\lambda_0^2}\Delta\lambda$$

Gives $B_{\odot}$ = 3.6 10^7 Wm^{-2} /2π sr

= 5.7 10^6 Wm^{-2} sr^{-1}

Compare this with the brightness of a 1mW helium-neon laser (λ=632nm) with a 0.5mm intra-cavity waist.

Divergence of beam $\theta_{1/2} \approx \dfrac{\lambda}{\pi\, w_0}$

Solid angle subtended by beam

$$\Omega = \pi\,\theta_{1/2}^2 \approx \frac{\lambda^2}{\pi\, w_0^2}$$

Giving brightness $B = \dfrac{P}{\Omega\,\pi w_0^2} = \dfrac{P}{\lambda^2}$

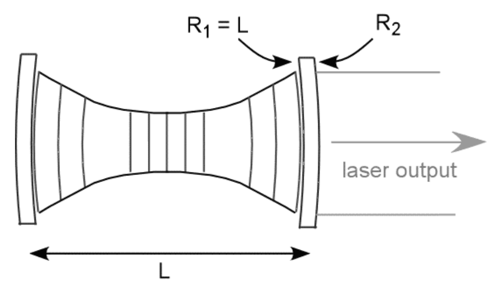
$$B = 2.5\times10^9\ Wm^{-2}sr^{-1}$$

Example

• A HeNe laser consists of a symmetric, confocal cavity 15cm long between the inner mirror surfaces.

The mirror substrates are made of glass with refractive index 1.5 and are 1cm thick. The outer surfaces are also curved, so that they can act as lenses to collimate the output beam (i.e. produce a waist at the laser output)

Find the radius of curvature required for the outer mirror surfaces and the diameter of the output beam.



Example

It is desired to mode-match a collimated laser beam into the lowest transverse mode of a symmetric confocal laser cavity of length 10cm using a suitable thin lens.

If the laser wavelength is 1μm, and the initial beam has a diameter 2mm, what is the required focal length of the lens?