3C43 Problem sheet 4 – Electro and non-linear optics

Solutions

Question 1

a) Explain what is the refractive index ellipsoid. [3]
b) Explain what is a uniaxial material. [3]
c) Write down the expression of the effective refractive for propagation of an electromagnetic wave in a uniaxial material with ordinary and extraordinary refractive indices n_o and n_e, respectively, if the propagation direction forms an angle theta with the principal axis z of the material. [3]
d) Calculate the refractive index in the case c above if n_o =1.5, if n_e =1.8, and theta = 45 degrees. [4]
e) Write down if the material (crystal) is a positive uniaxial or a negative uniaxial material. [2]

Solution

a) This is a construction enabling the determination of:
   1. The allowed polarisation directions for a given wave propagation direction
   2. The refractive indices experienced by the wave propagating in the specific anisotropic crystal or medium under consideration.

   The intersection of a plane perpendicular to the propagation direction with the index ellipsoid defines an ellipse whose major and minor axes are the directions of the allowed linear polarizations, and the lengths of these axes give the refractive indices.

b) A uniaxial materials is a material for which, for a given propagation direction, there are generally two refractive indices, corresponding to 2 polarization directions. There is one propagation direction in which the refractive index is independent of the polarisation: the optic axis. All components polarised perpendicular to the optic axis wee the same refractive index n_0. They are known as ordinary rays. Alternatively a uniaxial material can be defined as one having 2 principal refractive indices that are the same, and a third principal refractive index different from the first 2. A “principal” refractive index is the refractive index experienced by a wave propagating along one of the directions identified by the eigenvectors of the susceptibility tensor. These are special directions in which the polarisation P is in the same direction as the E field, even in the case of an anisotropic material.

c)

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

d) $$\frac{1}{n_e^2(\theta)} = \frac{\cos^2(\pi/4)}{1.5^2} + \frac{\sin^2(\pi/4)}{1.8^2} = \frac{1}{2}\left(\frac{1}{1.5^2} + \frac{1}{1.8^2}\right) = 0.376$$

$$n_e(\pi/4) = 1.6296$$

e) The material is a uniaxial positive one because $n_e > n_o$

Question 2

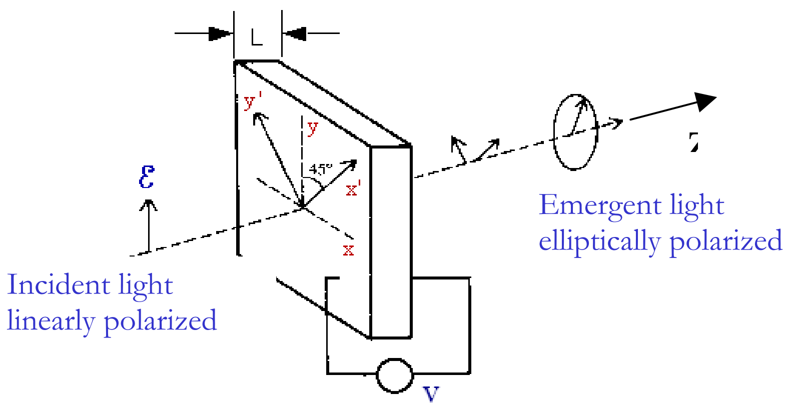
a) Write down a sketch of an electro-optic device for transforming a linearly polarised beam into either a circularly polarised or a 90 degrees rotated linearly polarised beam, and that makes use of a uniaxial anisotropic crystal. Explain the working principle of such a system. [3]

b) Define the half-wave voltage $V\pi$ and calculate its value for a crystal for which the only non-zero element of the susceptibility tensor is 1.4 pm/V, λ = 1.06 μm, and n_0 =1.6. [3]

c) Write down a sketch for a Pockels electro-optic modulator and show how this can be used for modulating the intensity of a linearly polarised laser beam. [4]

Solution

a)



Let us consider a "simple" electro-optic materials such as KDP, for which the variation of the index ellipsoid under the action of an electric field can be described by making use of just 1 electro-optic coefficient r. Let us also assume that the incoming beam is polarised along one of the principal (ordinary) axes of the electro-optic crystal, for example y, in the sketch. The sketch above shows such an electro-optic crystal of thickness L whose optical axis is aligned with the propagation direction of the beam under consideration. Under the action of the applied potential difference V and the related electric field D=V/L the intersection of the index ellipsoid with the plane normal to the propagation direction becomes an ellipsoid whose major and minor axis are rotated by 45 degrees (x' and y') with respect to the original principal axes of the index ellipsoid (x, y i.e. those in the absence of the electric field).
Since the mathematical description of the ellipsoid is now

$$x'^2\left(\frac{1}{n_o^2} + r\ \mathcal{E}_z\right) + y'^2\left(\frac{1}{n_o^2} - r\ \mathcal{E}_z\right) + \frac{z^2}{n_e^2} = 1$$

The refractive index ellipse axes are now $n_0+\Delta n$ and n0-Δn with

$$\Delta \frac{1}{n^2} = \frac{2}{n_o^3} \Delta n = r \quad E_z$$ and therefore

$$\Delta n = \frac{1}{2} n_0^3 \, r \quad E_z$$

After propagating a distance L in the crystal the phase difference induced between the x' and the y' linearly polarised components of the propagating beam is:

$$\Delta\varphi_{x'y'} = \frac{2\pi L}{\lambda} n_0^3 \, r \quad E \quad = \frac{2\pi}{\lambda} n_0^3 \, r \quad V$$

In general, the emerging light is elliptically polarised. For $\Delta\varphi = \pi$ the polarisation is rotated through 90 degrees and the corresponding voltage is called the half-wave voltage. For $\Delta\varphi = \pi/2$ the output polarisation is circular for linear polarisation at the input.

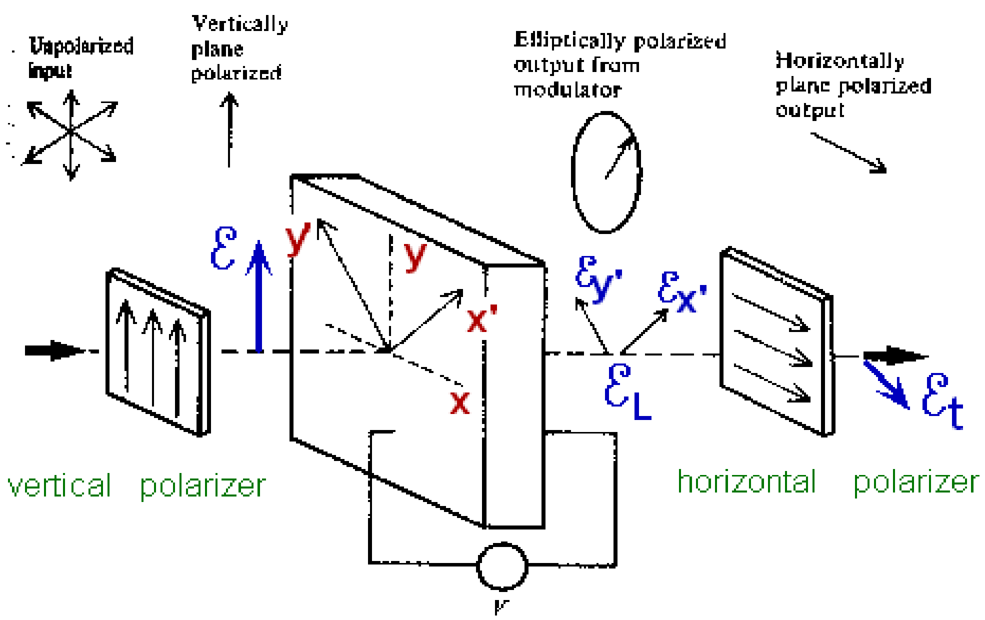
b) This is the voltage required to produce a phase difference of π between the components of the major and minor axis of the refractive index ellipse in uniaxial crystal under the action of an electric field.

$$V_\pi = \frac{\lambda}{2 n_0^3 \, r}$$

in our case, if r, the only non-zero element of the susceptibility tensor, is 1.4 pm/V, at 1.06 μm and $n_0 = 1.6$ we have

$$V_\pi = \frac{\lambda}{2 n_0^3 \, r} = \frac{1.06 \cdot 10^{-6}}{2 \cdot 1.6^3 \, 1.4 \cdot 10^{-12}} = 92.4 kV$$

c)



Field transmitted by input polariser (for unpolarised input light):

$$\vec{\mathcal{E}}_L = \frac{1}{\sqrt{2}}\mathcal{E}_0 \frac{y'+x'}{\sqrt{2}} \cos wt$$

Field after electro-optic crystal:

$$\vec{\mathcal{E}}_L = \frac{1}{\sqrt{2}}\mathcal{E}_0 \left\{\frac{y'}{\sqrt{2}} \cos\left(wt - \tfrac{1}{2}\Delta\phi\right) + \frac{x'}{\sqrt{2}} \cos\left(wt + \tfrac{1}{2}\Delta\phi\right)\right\}$$

$$\vec{\mathcal{E}}_L = \frac{1}{\sqrt{2}}\mathcal{E}_0 \left\{\frac{y-x}{2} \cos\left(wt - \tfrac{1}{2}\Delta\phi\right) + \frac{y+x}{2} \cos\left(wt + \tfrac{1}{2}\Delta\phi\right)\right\}$$

Field transmitted by output polariser (analyser):

$$\vec{\mathcal{E}}_t = \frac{\mathcal{E}_0\vec{x}}{2\sqrt{2}}\left\{\cos\left(wt + \tfrac{1}{2}\Delta\phi\right) - \cos\left(wt - \tfrac{1}{2}\Delta\phi\right)\right\} = -\frac{\mathcal{E}_0}{\sqrt{2}} \sin\frac{\Delta\varphi}{2} \sin\omega t$$

So intensity transmission

$$I_t / I_0 = \frac{1}{2}\sin^2\frac{\Delta\varphi}{2} = \frac{1}{2}\sin^2\frac{\pi}{2}\frac{V}{V_\pi}$$

where V_π is the half-wave voltage(and factor of ½ applies to incident unpolarised light, which becomes 1 for incident linearly polarised light).

Question 3

a) Write down an explanation of the context and the meaning of the so-called "slowly varying envelope approximation". [5]

b) After defining the meaning of the various symbols, show that solution of the non-linear wave equation leads to the following expression for the intensity of SHG

$$I_2(L) = \left|\chi^{(2)}\right|^2 I_1^2 \quad L^2 \ \frac{\sin^2(\Delta k\, L)}{(\Delta k\, L)^2}$$

under the following assumptions:

a. P is parallel to E
b. Incident plane wave propagates along the z axis
c. Only one frequency of the second order polarisation is important [5]

d) Plot the intensity of the SHG vs. the phase mismatch ΔkL [3]

c) Define the coherence length and calculate it for a crystal 10 cm long with $k_\omega = 122179\ \text{cm}^{-1}$ and $k_{2\omega} = 123137.3\ \text{cm}^{-1}$. [2]

Solution

a) The so-called "slowly varying envelope approximation" is encountered in the solution of the non-linear wave equation, for example in the case where second harmonic generation (SHG) is studied.

In this case, inserting $E_1 = \mathcal{E}_1 \ \exp i(\omega t - k_\omega z)$, the wave incident onto a non-linear crystal, and $E_2 = \mathcal{E}_2(z) \ \ \exp i(2\omega t - k_{2\omega} z)$ the generated second harmonic, into the non-linear wave equation one obtains:

$$\frac{\partial^2 E_1}{\partial z^2} + \frac{\partial^2 E_2}{\partial z^2} - \frac{n_\omega^2}{c^2}\frac{\partial^2 E_1}{\partial t^2} - \frac{n_{2\omega}^2}{c^2}\frac{\partial^2 E_2}{\partial t^2} = \mu_0 \varepsilon_0 \chi^{(2)} \frac{\partial^2 (E_1^2)}{\partial t^2}$$

and one of term that must be considered is thus:

$$\begin{aligned}\frac{\partial^2 E_2}{\partial z^2} &= -k_{2\omega}^2\ \mathcal{E}_2 \ \exp i(2\omega t - k_{2\omega} z) \\ &\quad - 2i\, k_{2\omega} \frac{\partial \mathcal{E}_2(z)}{\partial z} \exp i(2\omega t - k_{2\omega} z) \\ &\quad + \frac{\partial^2 \mathcal{E}_2(z)}{\partial z^2} \exp i(2\omega t - k_{2\omega} z)\end{aligned}$$

the mentioned slowly varying envelope approximation consists of disregarding the last term in the expression immediately above ($\frac{\partial^2 \mathcal{E}_2(z)}{\partial z^2}$). The rationale for doing so is that

$$\frac{\partial^2 \mathcal{E}_2(z)}{\partial z^2} << k_{2\omega} \frac{\partial \mathcal{E}_2(z)}{\partial z}$$

b)

- Assumptions:
  - P parallel to E, so can use a scalar equation
  - Incident plane-wave propagating along Oz
  - only one frequency of $P^{(2)}$ is important, the second-harmonic

Incident wave: $E_1 = \mathcal{E}_1 \exp i(\omega t - k_\omega z)$

Induced polarisation: $P^{(2)} = \varepsilon_0 \chi^{(2)} E_1^2$

$$= \varepsilon_0 \chi^{(2)} \mathcal{E}_1^2 \exp 2i(\omega t - k_\omega z)$$

Generated wave: $E_2 = \mathcal{E}_2(z) \exp i(2\omega t - k_{2\omega} z)$

Substituting in the non-linear wave-equation, writing for E the total electric field:

$$\frac{\partial^2 E_1}{\partial z^2} + \frac{\partial^2 E_2}{\partial z^2} - \frac{n_\omega^2}{c^2}\frac{\partial^2 E_1}{\partial t^2} - \frac{n_{2\omega}^2}{c^2}\frac{\partial^2 E_2}{\partial t^2} = \mu_0 \varepsilon_0 \chi^{(2)} \frac{\partial^2 (E_1^2)}{\partial t^2}$$

Work out all the terms:

$$\frac{\partial^2 E_1}{\partial z^2} = -k_\omega^2 \mathcal{E}_1 \exp i(\omega t - k_\omega z)$$

$$\frac{\partial^2 E_2}{\partial z^2} = -k_{2\omega}^2 \mathcal{E}_2 \exp i(2\omega t - k_{2\omega} z) - 2i\, k_{2\omega} \frac{\partial \mathcal{E}_2(z)}{\partial z} \exp i(2\omega t - k_{2\omega} z) + \cancel{\frac{\partial^2 \mathcal{E}_2(z)}{\partial z^2} \exp i(2\omega t - k_{2\omega} z)}$$

$$k_{2\omega} \frac{\partial \mathcal{E}_2(z)}{\partial z} >> \frac{\partial^2 \mathcal{E}_2(z)}{\partial z^2}$$

since $\mathcal{E}_2$ varies little on the scale of a wavelength (slowly-varying envelope approximation).

$$\frac{n_\omega^2}{c^2}\frac{\partial^2 E_1}{\partial t^2} = -k_\omega^2 \mathcal{E}_1 \exp i(\omega t - k_\omega z)$$

$$\frac{n_{2\omega}^2}{c^2}\frac{\partial^2 E_2}{\partial t^2} = -k_{2\omega}^2 \mathcal{E}_2(z) \exp i(2\omega t - k_{2\omega} z)$$

$$\mu_0 \frac{\partial^2 P^{(2)}}{\partial t^2} = \frac{-4\omega^2}{c^2} \chi^{(2)} \mathcal{E}_1^2 \exp 2i(\omega t - k_\omega z)$$

Substitute everything into the non-linear wave equation and compare coefficients of $\exp 2i\omega t$ on both sides of the equation:

$$\frac{\partial \mathcal{E}_2(z)}{\partial z} = -2i \frac{\omega^2 / c^2}{k_{2\omega}} \chi^{(2)} \mathcal{E}_1^2 \exp 2i\, \Delta k\, z$$

where $\Delta k = \cancel{\tfrac{1}{2}} k_{2\omega} - k_\omega$

For $\mathcal{E}_2(0) = 0$ the solution gives:

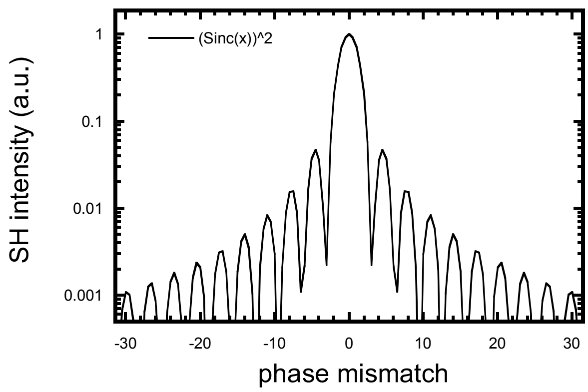
$$\mathcal{E}_2(L) = -i \frac{2\omega^2 / c^2}{k_{2\omega}} \chi^{(2)} \mathcal{E}_1^2 \exp(i\Delta k\, L)\; L \frac{\sin \Delta k\, L}{\Delta k\, L}$$

So the intensity of the second-harmonic wave is proportional to

$$I_2(L) = \left|\chi^{(2)}\right|^2 I_1^2\; L^2 \frac{\sin^2(\Delta k\, L)}{(\Delta k\, L)^2}$$

The coherence length, Lc, is given by $L_c = \pi / 2\Delta k$

The second-harmonic intensity increases for L up to L_c, then decreases.



c)

$$L_c = \pi / 2\Delta k = \pi / 2(123137 - 122179) = 1.64 * 10^{-3} \text{ cm}$$