ASTM-052 Extragalactic Astrophysics

ANSWERS: SET NUMBER 1

1. Kapteyn was unaware of the presence of interstellar dust in the plane of the Galaxy. (a) This limited visibility in the plane, so that only a fraction of it could be observed. (b) In the perpendicular direction, one can see right out of the plane. Kapteyn's model therefore had the true thickness of the plane but only a fraction of its diameter.

   We have

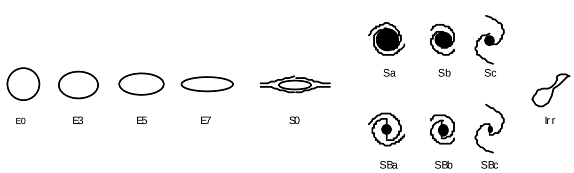
$$H_{\mathrm{o}} = h \times \frac{100 \times 10^{3} \left(\mathrm{m\,s^{-1}}\right)}{3.1 \times 10^{22} \left(\mathrm{m}\right)} = 3.2 \times 10^{-18}\,\mathrm{s}^{-1} \qquad (1.1)$$

   so that

$$\frac{1}{H_{\mathrm{o}}} = \frac{1}{h} \times \frac{10^{18}}{3.2 \times 3.2 \times 10^{7}}\, y \approx \frac{10^{10}}{h}\, y \qquad (1.2)$$

   Hubble measured h to be 5, giving H_{o}^{-1} – which is roughly the age of the universe – as 2×10^{9} y, less than the age, for example, of globular clusters in the Galaxy.

2.



   The flattest elliptical galaxies seen are E7, corresponding to the value 0.7 for e. From equation (2.1) of the question, we have

$$\cos\theta = 1 - e \qquad (2.1)$$

   so that

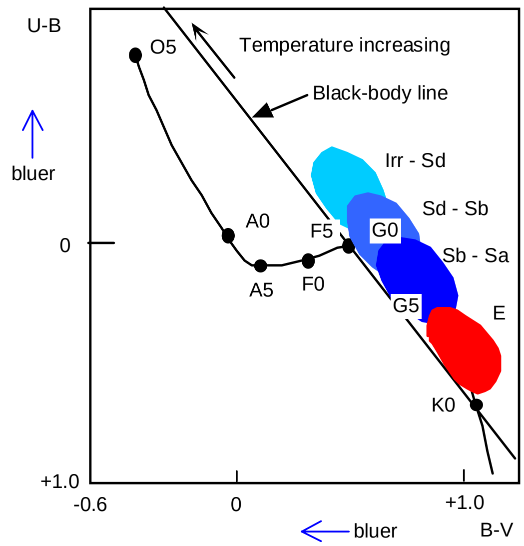
$$\left(\cos\theta\right)_{\mathrm{minimum}} = 0.3 \qquad (2.2)$$

   or

$$\theta_{\mathrm{maximum}} = \cos^{-1}\left(0.3\right) = 72.5° \qquad (2.3)$$

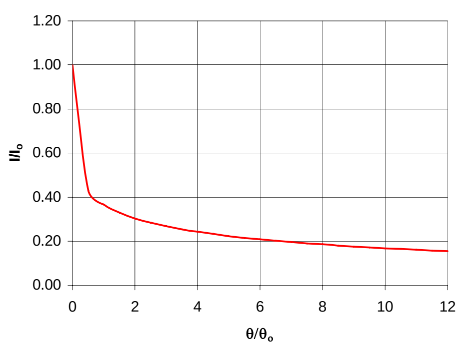
   [Remember that θ increases as $\cos\theta$ decreases.] Hence, if all elliptical galaxies were flat discs inclined at various angles to our line of sight, they would have to conspire to be so inclined by no more than about 75°.

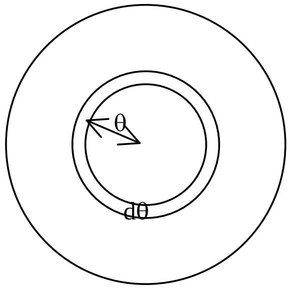
3. *A colour* is a measure of the ratio of the flux densities of a source at two different wavelengths.



   Hot young stars are blue whilst, on the whole, red stars are old. The figure shows that elliptical galaxies are redder than spirals and that spirals get bluer as one goes along the Hubble sequence. This suggest that ellipticals contain predominantly old stars whilst spirals contain increasingly more young stars as we go from *a* to *d*.

4. The surface brightness of a galaxy is its flux density per unit solid angle, as a function of position in the galaxian image.





Because $I(\theta)$ is given only as a function of θ, we can assume that the image of the galaxy has circular symmetry. Consider an annulus of (angular) radius θ and width $d\theta$. The solid angle $d\Omega$ subtended by this annulus is given by

$$d\Omega = 2\pi\theta d\theta \qquad (4.1)$$

and the flux dF coming from it is given by

$$dF = I(\theta)\times 2\pi\theta d\theta \; . \qquad (4.2)$$

The total flux density of the galaxy is therefore given by

$$F = \int_0^\infty I(\theta) 2\pi\theta\, d\theta \; . \qquad (4.3)$$

Substituting for the de Vaucouleurs profile in equation (4.3), we get

$$\begin{aligned} F &= \int_0^\infty I(0)\exp\left[-\left(\frac{\theta}{\theta_o}\right)^{1/4}\right]2\pi\theta\, d\theta \\ &= 2\pi I(0)\theta_o^2\int_0^\infty \exp\left[-\left(\frac{\theta}{\theta_o}\right)^{1/4}\right]\left(\frac{\theta}{\theta_o}\right)d\left(\frac{\theta}{\theta_o}\right) \end{aligned} . \qquad (4.4)$$

If we put

$$x = \left(\frac{\theta}{\theta_o}\right)^{1/4} , \qquad (4.5)$$

equation (4.4) becomes

$$\begin{aligned} F &= 2\pi I(0)\theta_o^2\int_0^\infty e^{-x}x^4 d\left(x^4\right) \\ &= 2\pi I(0)\theta_o^2 \times 4\int_0^\infty e^{-x}x^7\, dx = \pi I(0)\theta_o^2\times 8\times 7! . \end{aligned} \qquad (4.6)$$

from equation 43.4 given in the question. Hence,

$$F = 8!\pi\theta_o^2 I(0) \qquad (4.7)$$

From equation (4.3), we have for the fictitious galaxy,

$$\begin{aligned} F_{fict} &= \int_0^\infty I(\theta)2\pi\theta\, d\theta = 2\pi I_o \int_0^{\theta_{max}} \theta\, d\theta \\ &= 2\pi\theta_{max}^2 I_o \end{aligned} . \qquad (4.8)$$

If

$$I_o = I(0) \quad \text{and} \quad F_{fict} = F \qquad (4.9)$$

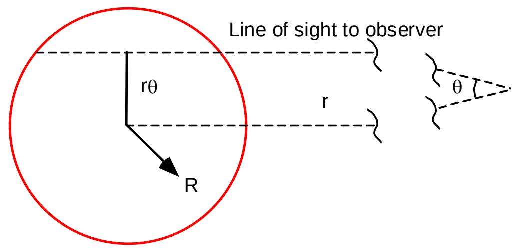
then we have from equations (4.7) and (4.8),

$$2\pi\theta_{max}^2 = 8!\pi\theta_o^2 \qquad (4.10)$$

so that

$$\theta_{max} = \sqrt{4\times 7!}\,\theta_o \approx 200\theta_o \; . \qquad (4.11)$$

This result shows what a very large contribution the outer regions of an elliptical galaxy make to the total luminosity.



The figure show schematically the relation between $I_E(\theta)$ and $J_E(R)$. Whilst $J_E(R)$ is a measure of the emission per unit volume as a function of distance from the centre of the galaxy, $I_E(\theta)$ is a measure of the emission per unit area of the image as a function of the projected (*angular*) distance from the centre. As can be seen from the sketch, $I_E(\theta)$ contains contribution from $J_E(R)$. at various value of R, In particular, at the θ=0 for example, the emission from the centre of the galaxy at R=0 is diluted by emission for regions, along the line of sight, further from the centre. This dilution means that $I_E(\theta)$ falls off less rapidly with θ than does $J_E(R)$ with R.

The same does not apply to the discs of spiral galaxies because they are flat and there is no integration along a significant line of sight.

5. Suppose that the mass M is related to G, R and v by

$$M = kG^\alpha R^\beta v^\gamma \, , \qquad (5.1)$$

where k is a dimensionless constant. In terms of dimensions, we have

$$\mathrm{M}^1 = \left[\mathrm{M}^{-1}\mathrm{L}^3\mathrm{t}^{-2}\right]^\alpha \mathrm{L}^\beta \left[\mathrm{L}\mathrm{t}^{-1}\right]^\gamma . \qquad (5.2)$$

Comparing coefficients of M, L and t on each side of the equation, we have

$$\begin{aligned} 1 &= -\alpha \\ 0 &= 3\alpha + \beta + \gamma \\ 0 &= -2\alpha - \gamma \end{aligned} \qquad (5.3)$$

having solutions

$$\begin{aligned} \alpha &= -1 \\ \beta &= 1 \\ \gamma &= 2 \end{aligned} \qquad (5.4)$$

so that

$$M = k \times G^{-1} R^{1} v^{2} \tag{5.5}$$

or

$$M \sim \frac{Rv^2}{G}. \tag{5.6}$$

6. We have for the width $\Delta\lambda_{\text{Dopper}}$ of the line arising from Doppler broadening,

$$\begin{aligned}\Delta\lambda_{\text{Doppler}} &= \sqrt{(0.503)^2 - (0.05)^2}\ \text{nm} \\ &= 0.500\ \text{nm}\end{aligned} \tag{6.1}$$

Hence, the mean-squared line-of-sight velocity $<v_r^2>$ is given by

$$\frac{\left\langle v_{\text{r}}^2 \right\rangle^{1/2}}{c} = \frac{\Delta\lambda}{\lambda_{\text{o}}} = \frac{0.5}{500} = 10^{-3} \tag{6.2}$$

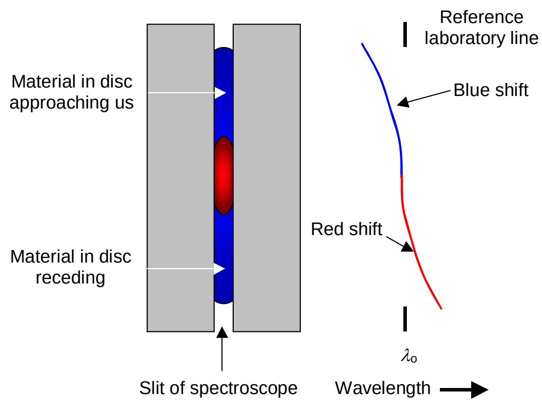
so that

$$\left\langle v_{\text{r}}^2 \right\rangle^{1/2} = 10^{-3} \times 3 \times 10^{8}\ \text{m s}^{-1} = 300\ \text{km s}^{-1}. \tag{6.3}$$

[Apologies for the typo in the question.]

If we assume that the mean-squared velocities are the same in each direction, then the total mean-squared velocity $<v^2>$ is $3<v_r^2>$.The mass Mcan be estimated from

$$\begin{aligned}M &\sim \frac{R\left\langle v^2 \right\rangle}{G} \\ &= \frac{50 \times 3.1 \times 10^{19} \times 3 \times \left(300 \times 10^{3}\right)^2}{6.67 \times 10^{-11}}\ \text{kg} \\ &= 3.6 \times 10^{42}\ \text{kg} \approx 2 \times 10^{12} \text{M}_{\text{sun}}.\end{aligned}$$



The figure shows schematically how long-slit spectra are used to measure a galaxy's rotation curve in the optical. The slit of the spectrograph is placed along the major axis of the galaxy as shown. The resultant spectrum of a single line of rest-wavelength λ_o is shown on the right, relative to the same line produced by a laboratory source. The parts of the galaxy that are approaching the observer give rise to blue shifted lines whilst those that are receding give rise to red-shifted lines. The overall effect is to produce the curved spectral line shown in the figure. Since distance along the slit is proportional to the distance from the centre of the galaxy, this can immediately be translated into a rotation curve.

The centripetal force needed keep material of mass m in circular orbit of radius r, with velocity $\Theta(r)$, about the centre of the galaxy is

$$-\frac{m\Theta^2(r)}{r} \tag{6.4}$$

This can only be supplied by the gravitational force:

$$-\frac{GM(r)}{r^2}, \tag{6.5}$$

where $M(r)$ is the mass contained within radius r of the centre. Equating the forces given in (6.4) and (6.5), we get

$$M(r) = \frac{r\Theta^2(r)}{G}. \tag{6.6}$$

Given that

$$\Theta(r) \approx \Theta_{\text{o}} \times \frac{r}{r_{\text{o}}}; \quad r \le r_{\text{o}} \tag{6.7}$$

in the inner part of the galaxy, we have from equation (4.3),

$$M(r) = \frac{r}{G} \times \left[\Theta_{\text{o}} \frac{r}{r_{\text{o}}}\right]^2 = \left(\frac{\Theta_{\text{o}}^2}{Gr_{\text{o}}^2}\right) r^3 \tag{6.8}$$

in these regions.

The mass $dM(r)$ contained in a thin shell of thickness dr at r is given by

$$dM(r) = 4\pi r^2 \rho(r) dr, \tag{6.9)]}$$

where r is the density at r. Hence

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dM(r)}{dr}. \tag{6.10)]}$$

Applying this to equation (4.7), we have

$$\begin{aligned}\rho(r) &= \frac{1}{4\pi r^2} \frac{d}{dr}\left[\left(\frac{\Theta_{\text{o}}^2}{Gr_{\text{o}}^2}\right) r^3\right] \\ &= \frac{3}{4\pi}\left(\frac{\Theta_{\text{o}}^2}{Gr_{\text{o}}^2}\right) = \text{constant}.\end{aligned} \tag{6.11)]}$$

By the definition of flux-density F, we have for the luminosity L of a galaxy that is at distance d,

$$L = \left(4\pi d^2\right) \times F \ .$$

Using the expression for d given in equation (7.4) of the question, we get

$$L = 4\pi \left(z \frac{c}{H_o} \right)^2 F \ . \tag{6.13}$$

If a galaxy of radius R is at a distance d, its angular diameter θ is given by

$$\theta = \frac{R}{d} \ . \tag{6.14}$$

If we substitute for R from equation (4.8) into equation (4.7) of the question, we get

$$M \sim (d \times \theta) \times \frac{v^2}{G} = \left(z \frac{c}{H_o} \right) \frac{v^2}{G} \theta \tag{6.15}$$

From equations (6.13) and (6.15), we have directly that

$$\frac{M}{L} \sim \frac{\left(z \frac{c}{H_o} \right) \frac{v^2}{G} \theta}{4\pi \left(z \frac{c}{H_o} \right)^2 F} = \left[\frac{1}{4\pi} \frac{v^2 \theta}{Gc \times zF} \right] H_o \ .$$

All the quantities in the square bracket of equation (6.16) are either constants or are directly observed. The measured mass-luminosity ratio is therefore directly proportional to the Hubble constant.

Elliptical galaxies have mass-luminosity ratios, measured in solar values, in the range of a few tens of h whereas the value for spirals is around 10 h.

7. The Euler equation applied to the density ρ of a fluid is

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{u}.\nabla)\rho \ . \tag{6.17}$$

The continuity equation is

$$0 = \frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \rho \nabla.\mathbf{u} + (\mathbf{u}.\nabla)\rho \ . \tag{6.18}$$

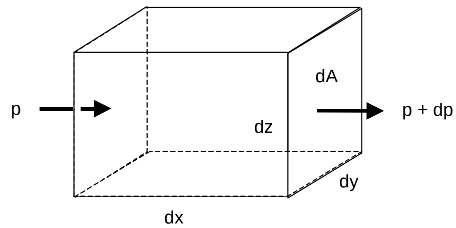
From equations (6.17) and (6.18), therefore, we get

$$\frac{D\rho}{Dt} = -\rho \nabla.\mathbf{u} \ . \tag{6.19}$$

For an *incompressible* fluid, $D\rho/Dt$ must be zero so

$$\nabla.\mathbf{u} = 0 \ . \tag{6.20}$$

8. (6.12)



Consider first the pressure acting on the faces of the a rectangular element of fluid parallel to the x-axis. From the figure, we have for the force df_{right} acting to the right on the left-hand face,

$$df_{\text{right}} = pdA \ , \tag{6.21}$$

where dA is the area of one of the cube's faces. The force df_{eft} acting to the left on the right-hand face is

$$df_{\text{left}} = (p + dp)dA \ . \tag{6.22}$$

The net pressure force df_p in the positive x-direction is therefore given by

$$\begin{aligned} df_x &= pdA - (p + dp)dA = -dpdA \\ &= -dp \times dydz. \end{aligned} \ . \tag{6.23}$$

(6.16)

The mass dm of fluid contained within the element is given by

$$dm = \rho dV = \rho \times dxdydz \tag{6.24}$$

so that the force F_x per unit mass of the fluid in the x-direction is given by

$$F_x dm = \frac{df_x}{dm} = \frac{-dp \times dxdy}{\rho dxdydz} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \ . \tag{6.25}$$

We get similar expressions for the y- and z-directions so that the total pressure force $\mathbf{F}_p$ per unit mass of the fluid is given by

$$\mathbf{F}_p = -\frac{1}{\rho} \nabla p \ . \tag{6.26}$$