UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. ***M.Sci.***

Physics 2B22: Quantum Physics

COURSE CODE	:	**PHYS2B22**
UNIT VALUE	:	0.50
DATE	:	**1 l-MAY-99**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours 30 minutes**

99-C1139-3- 140

TURN OVER

Answer SIX questions from Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

SECTION A

[Part marks]

1. Explain the terms **eigenvalue, eigenfunction** and **operator** as applied to quantum mechanical systems. [7]

2. Explain how quantum mechanics accounts for the photoelectric effect. [7]

3. Give the eigenvalues of the orbital angular momentum operators $\hat{L}^2$ and $\hat{L}_z$. [3]
   If the orbital angular momentum quantum number $\ell = 2$, sketch the possible orientations of the the angular momentum vector **L** in the semi-classical vector model. [4]

4. A particle executes one-dimensional simple harmonic motion. Sketch the classical and quantum-mechanical position probability distributions for such a particle with energies corresponding to quantum numbers $n = 1$ and $n = 10$. [4]
   Discuss the transition from the quantum to the classical regimes in this case. [3]

5. Consider a system in which an operator $\hat{P}$ represents some dynamical variable. What are the consequences if $\hat{P}$ commutes with the Hamiltonian operator, $\hat{H}$, for the system. [3]
   What is the additional consequence if $\hat{P}$ is an operator that does not depend explicitly on time? [4]

6. Sketch the effective potentials describing the motion of an electron in a hydrogen atom for orbital angular momentum quantum numbers $\ell = 0$ and 1. [3]
   Explain the form of the potential for small and large r in both cases. [4]

7. Neglecting spin, the bound states of the hydrogen atom are described by the quantum numbers n, ℓ, m_ℓ. Explain their significance and give their possible values. [4]
   Explain what is meant by **degeneracy** and illustrate your answer by considering the states of the hydrogen atom with n = 2. [3]

8. A dynamical variable represented by the linear operator $\hat{Q}$ has a complete set of orthonormal eigenstates ψ_i. If the expansion postulate is

$$\Psi = \sum_i c_i \psi_i,$$

show that

$$c_i = \int \psi_i^* \Psi d\tau$$

and give an interpretation of the complex numbers c_i.
An observable $\hat{Q}$ which has two orthonormal eigenstates ψ_+ and ψ_-, corresponding to eigenvalues +1 and -1 respectively, that form a complete set, is in a state Ψ such that

$$\Psi = \frac{3}{5}\psi_+ + \frac{4}{5}\psi_-.$$

What is the probability of finding on measurement the value of +1 for $\hat{Q}$?
What is the expectation value of $\hat{Q}$ in state Ψ ?

SECTION B

9. The operator representing the z-component of orbital angular momentum is

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}.$$

Determine the unnormalised eigenfunctions of $\hat{L}_z$.
Sketch the angular probability distribution associated with these eigenfunctions.
Show that ϕ and $\hat{L}_z$ satisfy the commutation relation

$$[\phi, \hat{L}_z] = i\hbar,$$

and explain the significance of this result in relation to the probability distribution obtained for the eigenfunctions of $\hat{L}_z$.
The simultaneous normalised eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ are the spherical harmonics $Y_{\ell,m}(\theta,\phi)$, where ℓ is the orbital angular momentum quantum number. For $\ell = 1$, the spherical harmonics are

$$Y_{1,0} = \text{J}\ \&\ \cos\theta$$
$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}.$$

Sketch the angular probability distributions associated with these three spherical harmonics.
Show that

$$|Y_{1,0}|^2 + |Y_{1,1}|^2 + |Y_{1,-1}|^2 = \frac{3}{4\pi},$$

and explain the significance of the absence of angular dependence in this result.

10. The one-dimensional time-independent Schrodinger equation for a particle of mass m moving in a potential $V(x)$ is given by

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)u(x) = Eu(x),$$

where E is the total energy. A potential barrier is defined by

Region 1	$x < 0$	and	$V(x) = 0$
Region 2	$0 \leq x \leq a$	and	$V(x) = V_0$
Region 3	$x > a$	and	$V(x) = 0$

where $V_0 > 0$ and $a > 0$.

(a) Consider the case where particles are incident on the barrier from Region 1 with total energy $E > V_0$. Demonstrate that the Schrödinger equation has the following solutions in the three regions,

$$\begin{aligned} u_1(x) &= e^{ikx} + Ae^{-ikx} \\ u_2(x) &= Be^{iqx} + Ce^{-iqx} \\ u_3(x) &= De^{ikx}, \end{aligned}$$

where $k^2 = \dfrac{2mE}{\hbar^2}$, $q^2 = \dfrac{2m(E-V_0)}{\hbar^2}$ and A, B, C and D are constants. [4]

(b) What is the significance of the two terms in the solution in Region 1 and why is there no term in e^{-ikx} for $x > a$? [3]

(c) State the continuity conditions that must be satisfied by the wave function at $x = 0$ and $x = a$. [3]

(d) In general, some particles are reflected by the barrier, but for particular values of the energy $E > V_0$, all particles are transmitted and none are reflected, so that $A = 0$. By using the continuity conditions at $x = 0$, show that when total transmission occurs,

$$B = \frac{(q+k)}{2q}, \qquad C = \frac{(q-k)}{2q}.$$

[4]

(e) Use the continuity conditions at $x = a$ to show that total transmission occurs only for energies for which $qa = n\pi$, where $n = 1, 2, 3, \ldots$ [6]

11. Including the finite mass of the nucleus: the radial Schrodinger equation for an electron in a hydrogen atom is, in atomic units,

$$\left(\frac{1}{\mu}\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{\mu r^2} + \frac{2}{r} + 2E\right) \mathrm{F(r)} = 0,$$

where E is the total energy, ℓ is the orbital angular momentum quantum number and μ is the reduced mass.

(a) Making the substitutions $\rho = \mu r$ and $E = -\mu/(2\nu^2)$, show that

$$\left(\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2}{\rho} - \frac{1}{\nu^2}\right) F(\rho) = 0.$$

(b) On putting $F(p) = \exp(-\rho/\nu)y(\rho)$, the radial Schrodinger equation becomes

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}\right] y = \frac{2}{\nu}\left[\frac{d}{d\rho} - \frac{\nu}{\rho}\right] y.$$

(c) Assuming that $y(\rho)$ can be expanded as the series

$$y(\rho) = \rho^{\ell+1} \sum_{p=0}^{\infty} a_p \rho^p,$$

where $a_0 \neq 0$, show that the coefficients a_p in the series satisfy the recurrence relation,

$$p(2\ell + p + 1)a_p = \frac{2}{\nu}(\ell + p - \nu)a_{p-1}.$$

for $p \geq 1$.

(d) Solutions of the radial Schrodinger equation exist which are bounded for all r provided that ν = n, where n is a positive integer. Show that the unnormalised radial function for the 1s state is

$$F_{1s}(r) = a_0\,(\mu r)\,\mathrm{e}^{-\mu r}.$$

If this function is normalised according to

$$\int_0^{\infty} F_{1s}^2(r)\mathrm{d}r = 1,$$

show that $a_0 = 2\sqrt{\mu}$.

(e) Obtain an expression for the mean radius of the 1s state.

The following results may be assumed

$$< r > = \int_0^{\infty} \mathrm{r}\ F^2(r)\mathrm{d}r \qquad\qquad \int_0^{\infty} r^m \mathrm{e}^{-\alpha r}\mathrm{d}r = \frac{m!}{\alpha^{m+1}}.$$

12. Briefly describe Compton's X-ray scattering experiment which demonstrated particle-like properties of electromagnetic radiation. [5]

A photon with frequency ν has an energy $h\nu$ and a momentum of magnitude $\gamma = h\nu/c$. An electron with momentum p has an energy

$$E = (m^2c^4 + p^2c^2)^{\frac{1}{2}},$$

where m is the electron rest mass.

In Compton scattering, a photon of frequency ν collides with an electron, whose momentum is assumed to be zero initially. After the collision, the photon has momentum $\gamma' = h\nu'/c$ and leaves at an angle θ to the direction of the incident photon. The electron momentum after the collision is p.

(a) Show that conservation of momentum leads to the relation

$$p^2 = (\gamma - \gamma')^2 + 2\gamma\gamma'(1 - \cos\theta).$$ [4]

(b) Show that conservation of energy can be expressed as

$$p^2 = (\gamma - \gamma')^2 + 2mc(\gamma - \gamma').$$ [4]

(c) Hence show that

$$\lambda' - \lambda = \lambda_C(1 - \cos\theta),$$

where λ and λ' are the photon wavelengths before and after the collision and $\lambda_C = \dfrac{h}{mc}$ is the Compton wavelength. [3]

(d) Show that when $\lambda \gg \lambda_C$, the ratio of the electron kinetic energy energy after the collision, T, to the initial photon energy is approximately

$$\frac{T}{h\nu} = \frac{\lambda_C}{\lambda}(1 - \cos\theta).$$ [4]

13. Show, on the basis of a simple classical model assuming a circular orbit, that a one-electron atom will have a magnetic moment M given by

$$\mathbf{M} = -\frac{e}{2m}\mathbf{L}$$

where e and m are the charge and mass of the electron and L is its orbital angular momentum vector. [7]

Describe with a diagram the Stern-Gerlach experiment to measure magnetic moments of atoms, and the results obtained. [7]

Why could the results obtained for Ag atoms not be explained either by classical mechanics, or by the above equation together with the quantum mechanical rules for the allowed values of orbital angular momentum ? [3]

Interpret the results in terms of electron spin. [3]

END OF PAPER