PHYS2B22 Quantum Physics
Evening course lecture notes. Set 5.
Sam Morgan 2005

Outline of section 5

Angular momentum in quantum mechanics

- Classical definition of angular momentum
- Linear Hermitian Operators for angular momentum
  Commutation relations
  Physical consequences
- Simultaneous eigenfunctions of total angular momentum and the z-component
  Vector model
- Spherical harmonics
  Orthonormality and completeness

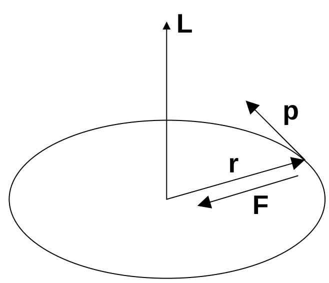
Classical angular momentum

For a **classical** particle, the angular momentum is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$
$$= L_x \mathbf{i} + L_y \mathbf{j} + L_z \mathbf{k}$$

In components

$$L_x = yp_z - zp_y$$
$$L_y = zp_x - xp_z$$
$$L_z = xp_y - yp_x$$



Same origin for **r** and **F**

Angular momentum is very important in problems involving a **central force** (one that is always directed towards or away from a central point) because in that case it is **conserved**

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$
$$= (\frac{\mathbf{p}}{m} \times \mathbf{p}) + (\mathbf{r} \times \mathbf{F}) = 0.$$

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Hermitian operators for quantum angular momentum

In quantum mechanics we get **linear Hermitian angular momentum operators** from the classical expressions using the postulates

$$\mathbf{r} \to \hat{\mathbf{r}} = \mathbf{r}, \quad \mathbf{p} \to \hat{\mathbf{p}} = -i\hbar\nabla$$
$$\mathbf{L} = \mathbf{r}\times\mathbf{p} \to \hat{\mathbf{L}} = -i\hbar\mathbf{r}\times\nabla$$

$$\begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x \end{aligned} \Rightarrow \begin{aligned} \hat{L}_x &= y\hat{p}_z - z\hat{p}_y \\ \hat{L}_y &= z\hat{p}_x - x\hat{p}_z \\ \hat{L}_z &= x\hat{p}_y - y\hat{p}_x \end{aligned} \Rightarrow \begin{aligned} \hat{L}_x &= -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] \\ \hat{L}_y &= -i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right] \\ \hat{L}_z &= -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] \end{aligned}$$

Commutation relations

The different components of angular momentum do not commute with one another, e.g.

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$

Proof: $\left[\hat{L}_x, \hat{L}_y\right] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x$

$$\begin{aligned} \hat{L}_x &= y\hat{p}_z - z\hat{p}_y \\ \hat{L}_y &= z\hat{p}_x - x\hat{p}_z \\ \hat{L}_z &= x\hat{p}_y - y\hat{p}_x \end{aligned}$$

$$[x, p_x] = i\hbar \quad [y, p_y] = i\hbar \quad [z, p_z] = i\hbar$$

Similar arguments give the cyclic permutations $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$ $\quad \left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$

Summarize these as $\left[\hat{L}_i, \hat{L}_j\right] = i\hbar\hat{L}_k$ where *i*, *j*, *k* obey a cyclic (*x*, *y*, *z*) relation

Commutation relations (2)

The different components of **L** do not commute with each another, but they do commute with the squared magnitude of the angular momentum vector:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

Proof: $\left[\hat{L}_x, \hat{L}^2\right] =$

Similar proofs for the other components

$$\left[\hat{L}_x, \hat{L}^2\right] = \left[\hat{L}_y, \hat{L}^2\right] = \left[\hat{L}_z, \hat{L}^2\right] = 0$$

Commutation relations (3)

The different components of angular momentum do not commute

- L_x, L_y and L_z are not compatible observables
- They do not have simultaneous eigenfunctions (except when **L** = **0**)
- We can not have perfect knowledge of any pair at the same time

BUT, the different components all commute with L^2

- L^2 and each component are compatible observables
- We can find simultaneous eigenfunctions of L^2 and one component

CONCLUSION

We can find simultaneous eigenfunctions of one component of angular momentum and *L^2*.
Conventionally we chose the ***z*** component. Next step is to find these eigenfunctions and study their properties.

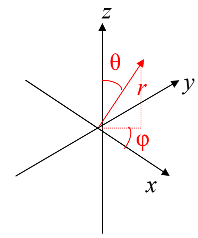
$$\hat{L}_z \psi = \alpha \hbar \psi$$
$$\hat{L}^2 \psi = \beta \hbar^2 \psi$$

What determines the direction of the z-axis?
In an experiment we usually have one or more privileged directions (e.g. the direction of an external electric or magnetic field) which gives a natural z axis. If not, this direction is purely arbitrary and no physical consequences depend on what choice we make.

Angular momentum in spherical polar coordinates

Spherical polar coordinates are the natural coordinate system in which to describe angular momentum: $x, y, z \rightarrow r, \theta, \phi$

The angular momentum operators only depend on the angles θ and φ and not on the radial coordinate r.



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\hat{L}_x = -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]$$
$$\hat{L}_y = -i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]$$
$$\hat{L}_z = -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

$\Rightarrow$

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Note: The angular momentum operators commute with any operator which only depends on r.
L^2 is closely related to the angular part of the Laplacian (see 2B72 and Section 6).

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}$$

L_z in spherical polars

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Proof that $L_z = -i\hbar\dfrac{\partial}{\partial\phi}$

$$\hat{L}_z = -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

$$\frac{\partial}{\partial\phi} = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z}$$

$$x = r\sin\theta\cos\phi \quad\Rightarrow\quad \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi = -y$$

$$y = r\sin\theta\sin\phi \quad\Rightarrow\quad \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi = x$$

$$z = r\cos\theta \quad\Rightarrow\quad \frac{\partial z}{\partial\phi} = 0$$

$$\frac{\partial}{\partial\phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{i}{\hbar}\hat{L}_z$$

Eigenfunctions of L_z

Look for simultaneous eigenfunctions of L^2 and L_z

First find the eigenvalues and eigenfunctions of L_z. Can only depend on the angle φ

$$\hat{L}_z \Phi_\alpha(\phi) = \alpha\hbar\Phi_\alpha(\phi)$$

$$-i\hbar\frac{\partial}{\partial\phi}\Phi_\alpha(\phi) = \alpha\hbar\Phi_\alpha(\phi)$$

$$\Rightarrow \Phi_\alpha(\phi) = A\exp(i\alpha\phi)$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Normalize solution

$$\int_0^{2\pi} d\phi \left|\Phi_\alpha(\phi)\right|^2 = 1$$

$$\left|\Phi_\alpha(\phi)\right|^2 = A^* \exp(-i\alpha\phi) A\exp(i\alpha\phi) = |A|^2$$

$$|A|^2 \int_0^{2\pi} d\phi = 2\pi|A|^2 = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2\pi}}$$

$$\Phi_\alpha(\phi) = \frac{1}{\sqrt{2\pi}}\exp(i\alpha\phi)$$

Eigenfunctions of L_z (2)

Boundary condition: wave-function must be single-valued

$$\Phi_\alpha(\phi) = \frac{1}{\sqrt{2\pi}}\exp(i\alpha\phi)$$

$$\Phi(\phi + 2\pi) = \Phi(\phi)$$

$$\Rightarrow \exp(i\alpha\phi) = \exp(i\alpha\phi)\exp(2\pi i\alpha)$$

$$\Rightarrow \exp(2\pi i\alpha) = 1$$

$$\Rightarrow \alpha = m = 0, \pm1, \pm2, \pm3\ldots$$

The angular momentum about the z-axis is quantized in units of hbar *(compare Bohr model).* The possible results of a measurement of L_z are

$$L_z = m\hbar$$
$$m = \text{integer}$$

So the eigenvalue equation and eigenfunction solution for L_z are

$$\hat{L}_z\Phi_m(\phi) = m\hbar\Phi_m(\phi)$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}\exp(im\phi)$$

Orthonormality and completeness

L_z is a Hermitian operator. Its eigenfunctions are orthonormal and complete for all functions of the angle φ that are periodic when φ increases by 2π.

$$\psi(\phi + 2\pi) = \psi(\phi)$$

Orthonormality

$$\int_0^{2\pi} \Phi_m^*(\phi)\Phi_n(\phi)d\phi = \frac{1}{2\pi}\int_0^{2\pi} \exp(i(n-m)\phi)d\phi = \delta_{mn}$$

Completeness

$$\psi(\phi) = \sum_{m=-\infty}^{\infty} a_m \Phi_m(\phi)$$

$$a_m = \int_0^{2\pi} \Phi_m^*(\phi)\psi(\phi)d\phi$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}\exp(im\phi)$$

Example

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}\exp(im\phi)$$

A particle has the angular wavefunction

$$\psi(\phi) = \sqrt{\frac{1}{3\pi}}(1 + i\cos 3\phi), \quad 0 \le \phi \le 2\pi$$

Find, by inspection or otherwise, the coefficients a_m in the expansion

$$\psi(\phi) = \sum_{m=-\infty}^{\infty} a_m \Phi_m(\phi)$$

Hence confirm that the wavefunction is normalized.

What are the possible results of a measurement of L_z and their corresponding probabilities?

Hence find the expectation value of L_z for many such measurements on identical particles.

Eigenfunctions of L²

Now look for eigenfunctions of L^2

$$\hat{L}^2 f(\theta,\phi) = \beta\hbar^2 f(\theta,\phi)$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Try a separated solution of the form

$$f(\theta,\phi) = \Theta(\theta)\Phi(\phi) = \frac{1}{\sqrt{2\pi}}\exp(im\phi)\Theta(\theta)$$

(this ensures the solutions remain eigenfunctions of L_z)

Eigenvalue equation is

$$\hat{L}^2\left[\exp(im\phi)\Theta(\theta)\right] = \beta\hbar^2\exp(im\phi)\Theta(\theta)$$

We get the equation for $\Theta_{\beta m}(\theta)$ which depends on both β and m

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}m^2+\beta\right]\Theta_{\beta m}(\theta)=0$$

Eigenfunctions of L² (2)

Make the substitution

$$\mu = \cos\theta \Rightarrow \frac{\partial}{\partial\theta} = \frac{d\mu}{d\theta}\frac{\partial}{\partial\mu} = -\sin\theta\frac{\partial}{\partial\mu}$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}m^2+\beta\right]\Theta_{\beta m}(\theta)=0$$

This gives the **Legendre equation**, solved in 2B72 by the Frobenius method.

$$\frac{d}{\partial\mu}\left[\left(1-\mu^2\right)\frac{d\Theta_{\beta m}(\mu)}{\partial\mu}\right]+\left[\beta-\frac{m^2}{\left(1-\mu^2\right)}\right]\Theta_{\beta m}(\mu)=0$$

We need solutions that are finite at $\mu = \pm 1$ (i.e. at $\theta = 0$ and $\theta = \pi$ since $\mu = \cos\theta$).
This is only possible if β satisfies

$$\beta = l(l+1) \text{ where } l = 0,1,2,\ldots \text{ and } l \geq |m|$$

This is like the SHO where we found restrictions on the energy eigenvalue in order to produce normalizable solutions.

Eigenfunctions of L^2 (3)

Label solutions to the **Legendre equation** by the values of l and m

$$\Theta_{\beta m}(\mu) \rightarrow \Theta_{lm}(\mu)$$
$$\beta = l(l+1)$$

$$\frac{d}{\partial\mu}\left[\left(1-\mu^2\right)\frac{d\Theta_{lm}(\mu)}{\partial\mu}\right]+\left[l(l+1)-\frac{m^2}{\left(1-\mu^2\right)}\right]\Theta_{lm}(\mu)=0$$

For m = 0 the finite solutions are the **Legendre polynomials**

$$\Theta_{l,m=0}(\mu) = P_l(\mu) = P_l(\cos\theta)$$

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu$$
$$P_2(\mu) = \frac{1}{2}\left(3\mu^2 - 1\right)$$

For non-zero m the solutions are the **associated Legendre polynomials**

$$\Theta_{lm}(\mu) = P_l^m(\mu) = P_l^m(\cos\theta)$$

$$P_l^m(\mu) = (1-\mu^2)^{|m|/2}\left(\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^{|m|} P_l(\mu)$$

Note that these only depend on the size of m not on its sign

Eigenvalues of L^2

So the eigenvalues of L^2 for physically allowed solutions are

$$\beta\hbar^2 = l(l+1)\hbar^2, \text{ where } l = 0,1,2,\ldots \text{ and } l \geq |m|$$

The possible results for the measurement of the squared magnitude of the angular momentum are $L^2 = \beta\hbar^2 = l(l+1)\hbar^2$

The possible results for a measurement of the magnitude of the angular momentum are $|\mathbf{L}| = \sqrt{l(l+1)}\hbar$

$$\text{From } l \geq |m| \text{ we get } -l \leq m \leq l$$

For each l there are 2l+1 possible integer values of m

The restriction on the possible values of m is reasonable. The z-component of angular momentum can not be greater than the total!
In fact, unless l = 0, the z-component is always less than the total and can never be equal to it. Why?

Summary

The simultaneous eigenfunctions of L_z and L^2 are

$$f(\theta,\phi) \propto P_l^m\left(\cos\theta\right)\exp\left(im\phi\right)$$

Eigenvalues of $\hat{L}^2$ are $l(l+1)\hbar^2$, with $l = 0,1,2,\ldots$

The integer *l* is known as the **principal angular momentum quantum number**. It determines the **magnitude** of the angular momentum

Eigenvalues of $\hat{L}_z$ are $m\hbar$, with $-l \le m \le l$
(i.e. $m = -l, -l+1\ldots, -1, 0, 1, \ldots l-1, +l$)

The integer *m* is known as the **magnetic quantum number**. It determines the **z-component** of angular momentum. For each value of l there are **2l+1 possible values** of m.

The simultaneous eigenfunctions of L^2 and L_z do not correspond to definite values of L_x and L_y, because these operators do not commute with L_z. We can show, however, that the **expectation value** of L_x and L_y is zero for the functions f(θ,φ).

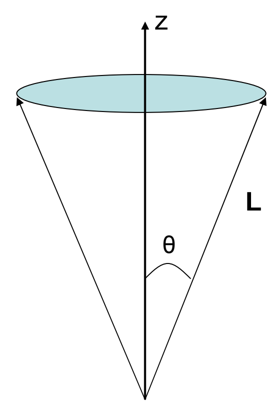
The vector model

This is a useful semi-classical model of the quantum results.

Imagine L precesses around the z-axis. Hence the magnitude of **L** and the z-component L_z are constant while the x and y components can take a range of values and average to zero, just like the quantum eigenfunctions.

A given quantum number l determines the magnitude of the vector **L** via

$$L^2 = l(l+1)\hbar^2$$
$$\left|\mathbf{L}\right| = \sqrt{l(l+1)}\hbar$$



The z-component can have the 2l+1 values corresponding to

$$L_z = m\hbar, \quad -l \le m \le l$$

In the vector model this means that only particular special angles between the angular momentum vector and the z-axis are allowed

The vector model (2)

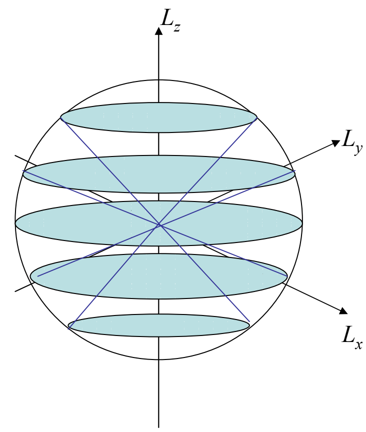
Example: l=2

Magnitude of the angular momentum is

$$L^2 = l(l+1)\hbar^2 = 6\hbar^2$$
$$|\mathbf{L}| = \sqrt{l(l+1)}\hbar = \sqrt{6}\hbar$$

Component of angular momentum in *z*- direction can be

$$-l \le m \le l \Rightarrow L_z = -2\hbar,\ -\hbar,\ 0,\ \hbar,\ 2\hbar$$



Quantum eigenfunctions correspond to a cone of solutions for **L** in the vector model

Spherical harmonics

The simultaneous eigenfunctions of L^2 and L_z are usually written in terms of the **spherical harmonics**

$$f(\theta,\phi) \rightarrow Y_{lm}(\theta,\phi) = N_{lm} P_l^m(\cos\theta)\exp(im\phi)$$

$$\hat{L}_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi)$$
$$\hat{L}^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$$

Proportionality constant N_{lm} is chosen to ensure normalization

NB. Some books write the spherical harmonics as $Y_l^m(\theta,\phi)$

Remember
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

First few examples (see 2B72):

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$
$$Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\exp(i\phi) = -\sqrt{\frac{3}{8\pi}}\frac{(x+iy)}{r}$$
$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$
$$Y_{1-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\exp(-i\phi) = \sqrt{\frac{3}{8\pi}}\frac{(x-iy)}{r}$$
$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

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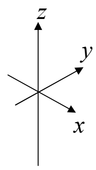
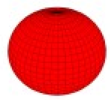
Shapes of the spherical harmonics



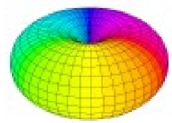
Y_{00}



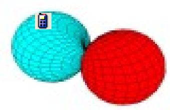
$l=0,\ m=0$

$$Y_{00}=\frac{1}{\sqrt{4\pi}}$$

Y_{11}



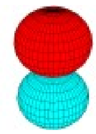
$\mathrm{Re}[Y_{11}]$



$l=1,\ m=1$

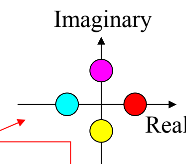
$$Y_{11}=-\sqrt{\frac{3}{8\pi}}\sin\theta\exp(i\phi)$$

Y_{10}



$l=1,\ m=0$

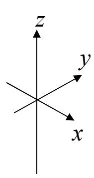
$$Y_{10}=\sqrt{\frac{3}{4\pi}}\cos\theta$$



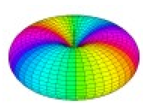
To read plots: distance from origin corresponds to magnitude (modulus) of plotted quantity; colour corresponds to phase (argument).

(Images from http://odin.math.nau.edu/~jws/dpgraph/Yellm.html)

Shapes of spherical harmonics (2)



Y_{22}

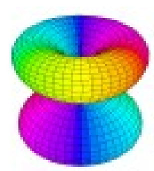


$\mathrm{Re}[Y_{22}]$

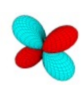
$l=2,\ m=2$

$$Y_{22}=\sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(2i\phi)$$

Y_{21}



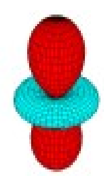
$\mathrm{Re}[Y_{21}]$



$l=2,\ m=1$

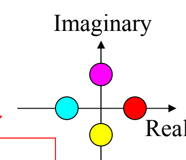
$$Y_{21}=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\exp(i\phi)$$

Y_{20}



$l=2,\ m=0$

$$Y_{20}=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$$



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(Images from http://odin.math.nau.edu/~jws/dpgraph/Yellm.html)

Orthonormality of spherical harmonics

The spherical harmonics are eigenfunctions of Hermitian operators. Solutions for different eigenvalues are therefore automatically orthogonal when integrated over all angles (i.e. over the surface of the unit sphere). They are also normalized so they are **orthonormal**.

Integration is over the solid angle

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

which comes from

$$d^3\mathbf{r} = r^2 dr d\Omega = r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y_{lm}^*(\theta,\phi) Y_{l'm'}(\theta,\phi) = 1 \quad \text{if } l = l' \text{ and } m = m'$$
$$= 0 \quad \text{otherwise}$$

Convenient shorthand

$$\int d\Omega \; Y_{lm}^*(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'} \delta_{mm'}$$

Compare: 1D Cartesian version of orthonormality

$$\int_{-\infty}^{\infty} \phi_n^*(x) \phi_m(x) dx = \delta_{mn}$$

Completeness of spherical harmonics

The spherical harmonics are a complete, orthonormal set for functions of two angles. Any function of the two angles θ and φ can be written as a linear superposition of the spherical harmonics.

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi)$$

$$-l \le m \le l$$

Using orthonormality we can show that the expansion coefficients are

$$a_{lm} = \int d\Omega \; Y_{lm}^*(\theta,\phi) f(\theta,\phi)$$
$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y_{lm}^*(\theta,\phi) f(\theta,\phi)$$

Compare: 1D version

$$\psi(x) = \sum_n a_n \phi_n(x)$$

$$a_n = \int_{-\infty}^{\infty} \phi_n^*(x) \psi(x) dx$$

Examples

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}Y_{lm}(\theta,\phi)$$

1) A particle has the un-normalized angular wavefunction

$$\psi(\theta,\phi)=\sqrt{\frac{4}{5}}Y_{00}+\sqrt{\frac{3}{5}}Y_{11}+\sqrt{\frac{2}{5}}Y_{10}+\sqrt{\frac{6}{5}}Y_{1-1}+Y_{21}$$

a) Normalize this wavefunction.

b) What are the possible results of a measurement of L_z and their corresponding probabilities? What is the expectation value of many such measurements?

c) What are the possible results of a measurement of L^2 and their corresponding probabilities? What is the expectation value of many such measurements?

Examples (2)

$$Y_{11}(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\exp(i\phi)$$

$$Y_{10}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1-1}(\theta,\phi)=\sqrt{\frac{3}{8\pi}}\sin\theta\exp(-i\phi)$$

1) A particle has normalized angular wavefunction

$$\psi(\theta,\phi)=\sqrt{\frac{15}{14\pi}}\cos 2\theta\sin\phi$$

Find the probability of measuring $L^2=2\hbar^2$

You can use the result $\int_0^{\pi}\cos 2\theta\sin^2\theta d\theta=-\pi/4$

Summary

The simultaneous eigenfunctions of L_z and L^2 are the spherical harmonics $Y_{lm}(\theta,\phi)$

$$\hat{L}_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi)$$
$$\hat{L}^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$$

Eigenvalues of $\hat{L}^2$ are $l(l+1)\hbar^2$, with $l = 0,1,2,\ldots$

l = principal angular momentum quantum number.

Determines the magnitude of the angular momentum.

Eigenvalues of $\hat{L}_z$ are $m\hbar$, with $-l \le m \le l$

m = magnetic quantum number.

Determines the z-component of angular momentum.

The spherical harmonics are a complete orthonormal set for functions of two angles

$$\int d\Omega\; Y_{lm}^*(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega \equiv \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta$$

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi)$$

$$a_{lm} = \int d\Omega\; Y_{lm}^*(\theta,\phi) f(\theta,\phi)$$