Physics 3C26 - Problem Set 1

December 21, 2005

TO BE HANDED IN FEBRUARY 2

1. **Quick questions [20]**

   (a) In classical mechanics we determine positions and momenta of all particles for all times. What do we determine from quantum mechanics? [2]

   (b) What is the meaning of the wave function? Why does $\int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x = 1$? [2]

   (c) If you have a solution of time independent Schrödinger equation with eigenfunction $\Phi_n(x)$ and energy E_n, what does the corresponding time-dependent wavefunction $\Psi(xt)$ equal? [2]

   (d) What is the meaning of a quantum expectation value? [2]

   (e) What property of Hermitian operators makes them suitable for describing physical observables? [2]

   (f) If I expand a wave function $\psi(x)$ in terms the eigenfunctions $\phi_n(x)$ of operator $\hat{Q}$ with corresponding eigenvalues q_n, so that $\psi(x) = \sum_n C_n \phi_n(x)$, what is the expectation value of $\hat{Q}$? What is the meaning of the expansion coefficients C_n? [2]

   (g) What does the expression $[\hat{A}, \hat{B}]$ equal? What is its name? [2]

   (h) If $[\hat{A}, \hat{B}] = 0$, what can you say about the eigenfunctions of $\hat{A}$ and $\hat{B}$? [2]

   (i) In Dirac notation, what does $|n\rangle$ correspond to? What is meant by $\langle n|\hat{Q}|m\rangle$? [2]

   (j) In matrix notation, how do we represent wave functions? How do we represent operators? [2]

2. **Wavefunctions [20]**

Consider a particle in one dimension whose wave function is $\psi(x) = Nx\exp(-\alpha x^2/2)$.

(a) Show that $N = \left(4\alpha^3/\pi\right)^{1/4}$[5]

(b) Write down an expression for the expectation value of the position of the particle and show $\langle x \rangle = 0$. What is the probability of finding the particle at in a very narrow region of width Δx about this position? [5]

(c) Write the expression for the mean uncertainty in the position of the particle, and show that it equals $\sigma_x = \sqrt{3/2\alpha}$. [5]

(d) Sketch a graph of $\psi(x)$ for both positive and negative x and mark on it the positions of $\langle x \rangle$ and $\pm\sigma_x$. [5]

You can use

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2}\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{\infty} x^2\mathrm{e}^{-\alpha x^2}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}, \quad \int_{-\infty}^{\infty} x^4\mathrm{e}^{-\alpha x^2}\mathrm{d}x = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$$

3. **Dirac notation [20]**

(a) Consider a set of orthonormal states represented in Dirac notation by $|n\rangle$. What does $\langle n|n\rangle$ equal? What does $\langle n|m\rangle$ ($n \neq m$) equal? [2]

(b) Suppose that $|n\rangle$ is an eigenstate of the Hamiltonian $\hat{H}$ so that $\hat{H}|n\rangle = \epsilon_n|n\rangle$. Using this equation, what does $\langle n|\hat{H}|n\rangle$ equal? What does $\langle n|\hat{H}|m\rangle$ ($n \neq m$) equal? [2]

(c) Now suppose we create a general state $|\psi\rangle = \sum_n C_n|n\rangle$. What does $\langle\psi|\hat{H}|\psi\rangle$ equal? [4]

(d) Write down the expression for the variance of the measured energy as an expectation value of the form $\langle\psi|\ldots|\psi\rangle$. What does the variance equal? [4]

(e) We would like to write the operator $\hat{H}$ in Dirac notation. That is: $\hat{H} = \sum_{nm} a_{nm}|n\rangle\langle m|$. By using the equation $\hat{H}|n\rangle = \epsilon_n|n\rangle$, find a_{nm}. [4]

(f) Consider two states $|a\rangle$ and $|b\rangle$. Now let $\hat{H}$ operate on these to give $\hat{H}|a\rangle = |A\rangle$ and $\hat{H}|b\rangle = |B\rangle$. What are the corresponding equations for $\hat{H}^\dagger$? If $\hat{H}$ is hermitian, show that $\langle A|b\rangle = \langle a|B\rangle$. Hence show that the eigenvalues of $\hat{H}$ are real. [4]