Physics 3C26 - Problem Set 2

January 4, 2006

TO BE HANDED IN FEBRUARY 16

1. **Quick questions [20]**

   (a) What are the allowed energies for a quantum harmonic oscillator with angular frequency ω? [2]

   (b) What classical property of an oscillator increases as the energy increases? [2]

   (c) Right down the raising and lowering operators for a harmonic oscillator in terms of the momentum and position operators. [2]

   (d) If $|n\rangle$ is an eigenvector for the harmonic oscillator, what does $\hat{a}_+|n\rangle$ equal? [2]

   (e) Write down the classical and quantum expressions for L_x (the x component of angular momentum). [2]

   (f) What does $[\hat{L}^2, \hat{L}_x]$ equal? What are the eigenvalues of $\hat{L}^2$? [2]

   (g) How are the angular momentum raising and lowering operators written in terms of $\hat{L}_x$ and $\hat{L}_y$? [2]

   (h) If $\hat{L}_z|lm\rangle = m\hbar|lm\rangle$, what are the largest and smallest values of m? [2]

   (i) What are the allowed eigenvalues for the electron spin operators $\hat{S}^2$ and $\hat{S}_z$? [2]

   (j) What are the central relations for general angular momentum operators that allow us to obtain all properties of angular momentum? [2]

2. **Harmonic Oscillator [20]**

(a) By using $[\hat{p}, \hat{x}] = -\mathrm{i}\hbar$ and $[\hat{p}, \hat{p}] = [\hat{x}, \hat{x}] = 0$ show that $[\hat{a}_-, \hat{a}_+] = 1$. [4]

(b) Starting from $\hat{a}_-|n\rangle = c_{n-1}|n-1\rangle$ and using $\langle n-1|n-1\rangle = 1$ and the definition of the number operator $\hat{N} = \hat{a}_+\hat{a}_-$, show that $c_{n-1} = -\mathrm{i}\sqrt{n}$. [4]

(c) Using the raising and lowering operators, show that $\langle n|\frac{1}{2}m\omega^2\hat{x}^2|n\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$. [4]

(d) Similarly, show that $\langle n|\frac{1}{2m}\hat{p}^2|n\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$. [4]

(e) What does $\langle n|\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2|n\rangle$ equal? Why would you expect this? [4]

3. **Angular Momentum [20]**

(a) Using the expression for $\hat{L}$ in terms of spherical coordinates,

$$\hat{L} = -\mathrm{i}\hbar \begin{pmatrix} -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial\phi} \end{pmatrix}$$

show that $\hat{L}_\pm = \hbar e^{\pm\mathrm{i}\phi}\left(\mathrm{i}\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right)$. [5]

(b) Consider $Y_{00} = \frac{1}{\sqrt{4\pi}}$. Using the raising and lowering operators you have just derived show that $Y_{01} = Y_{0-1} = 0$. Why is this not surprising? [5]

(c) Using $\hat{L}_+ = \hbar\sqrt{(l-m)(l+m+1)}|lm+1\rangle$ and $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$ show that $Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{\mathrm{i}\phi}$. [5]

(d) Using $\hat{L}_- = \hbar\sqrt{(l+m)(l-m+1)}|lm-1\rangle$ and $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$ show that $Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-\mathrm{i}\phi}$. [5]