Physics 3C26 - Problem Set 4

March 8, 2006

TO BE HANDED IN MARCH 16

1. **Quick questions [20]**

   (a) What is perturbation theory for? [2]

   (b) If the Hamiltonian for the system is $\hat{H} = \hat{H}_0 + \lambda\hat{V}$, where $\hat{H}_0\phi_n = E_n\phi_n$ and $\hat{H}\psi_n = W_n\psi_n$, write down the starting expressions for ψ_n and W_n used in perturbation theory. [2]

   (c) What functions do we use to expand the corrections to the wavefunction? [2]

   (d) Write down the first order correction to the energy. [2]

   (e) Write down the first order correction to the wavefunction. [2]

   (f) Write down the second order correction to the energy. [2]

   (g) Why do degenerate states need to be treated in a special way? [2]

   (h) What is the first order correction to the energy of degenerate states? [2]

   (i) What kind of perturbation theory do we need to used to describe the linear Stark effect for hydrogen? [2]

   (j) Where does the degeneracy come from in positronium? [2]

2. **Degenerate perturbation theory [20]**

   We will take another look at positronium, but proceed differently. The perturbation is $\hat{V} = A\hat{s}_e \cdot \hat{s}_p$. There are four degenerate spin states: $|\alpha_e\alpha_p\rangle$, $|\alpha_e\beta_p\rangle$, $|\beta_e\alpha_p\rangle$ and $|\beta_e\beta_p\rangle$.

   (a) Show that we can write $\hat{s}_e \cdot \hat{s}_p = \hat{s}_{ex}\hat{s}_{px} + \hat{s}_{ey}\hat{s}_{py} + \hat{s}_{ez}\hat{s}_{pz}$ as $\hat{s}_e \cdot \hat{s}_p = \frac{1}{2}\left(\hat{s}_{e+}\hat{s}_{p-} + \hat{s}_{e-}\hat{s}_{p+}\right) + \hat{s}_{ez}\hat{s}_{pz}$ where $\hat{s}_{e\pm} = (\hat{s}_{ex} \pm \mathrm{i}\hat{s}_{ey})$ and $\hat{s}_{p\pm} = (\hat{s}_{px} \pm \mathrm{i}\hat{s}_{py})$. [4]

   (b) Using the fact that $\hat{s}_{e+}$ and $\hat{s}_{p+}$ are raising operators and that $\hat{s}_{e-}$ and $\hat{s}_{p-}$ are lowering operators show that [10]

$$\langle m_e m_p|\hat{s}_e \cdot \hat{s}_p|m_e' m_p'\rangle = \hbar^2 \begin{pmatrix} -1/4 & 1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}$$

(c) We now calculate the energy corrections from this matrix by finding its eigenvalues. As it is block diagonal we can treat the blocks one at a time. The bottom right block is already diagonal, and has two eigenvalues both equal to $\hbar^2/4$. Thus the last two energy corrections are $A\hbar^2/4$. Now take the top left 2×2 block and show that its eigenvalues are $\hbar^2/4$ and $-3\hbar^2/4$, and write down the corresponding energy shifts. [6]

3. **Non-degenerate perturbation theory [20]**

We will look at a square well with a corrugated perturbing potential (it could be a simple model of a one-dimensional crystal). The wave functions ψ_n and energies ε_n for the infinite square well are

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L}) & 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases} \qquad \varepsilon_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

The perturbing potential is $V(x) = V_0 \sin(\frac{m\pi x}{L})$, where m is odd. In the following questions you can use

$$\sin A \sin B \sin C = \frac{1}{4}\left[\sin(A+B-C) + \sin(A-B+C) - \sin(A+B+C) - \sin(A-B-C)\right]$$

(a) Show that the first order change in energy for state n is $\Delta\varepsilon_n^{(1)} = \frac{8V_0}{\pi}\frac{n^2}{m(4n^2-m^2)}$. [7]

(b) Show that the matrix element $V_{nn'} = \int_{-\infty}^{\infty} \psi_n(x)V(x)\psi_{n'}(x)\,\mathrm{d}x$ is given by

$$V_{nn'} = -\frac{V_0}{2\pi}\left((-1)^{(n-n')}+1\right)\left[\frac{1}{n+m-n'} + \frac{1}{n-m+n'} - \frac{1}{n-m-n'} - \frac{1}{n+m+n'}\right]$$

[9]

(c) Now show that the first order correction to the wave function ψ_n is

$$\Delta\psi_n^{(1)} = \frac{V_0}{2\pi}\frac{2m}{\hbar^2}\left(\frac{L}{\pi}\right)^2\sqrt{\frac{2}{L}}\sum_{p\neq n}\frac{(-1)^{p-n}+1}{n^2-p^2}\left[\frac{1}{p+m-n} + \frac{1}{p-m+n} - \frac{1}{p-m-n} - \frac{1}{p+m+n}\right]\sin(\frac{p\pi x}{L})$$

[4]