Physics 3C26 - Solutions to Problem Set 1

December 21, 2005

1. **Quick questions [20]**

   (a) The probabilities of measuring values of observables [2]

   (b) The modulus squared of the wavefunction gives the probability density for finding a particle at a given point. $\int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x = 1$ because the probabilities must add up to 1. [2]

   (c) $\Psi(xt) = \Phi_n(x) \exp(E_n t/\mathrm{i}\hbar)$ [2]

   (d) A quantum expectation value is the average of some observable obtained by measuring its value many times for system that have been prepared in the same way for each experiment. [2]

   (e) Hermitian operators have real eigenvalues. [2]

   (f) $\langle Q \rangle = \sum_n |C_n|^2 q_n$. $|C_n|^2$ is the probability that when the observable Q is measured the value q_n is obtained. [2]

   (g) $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. It is a commutator. [2]

   (h) If $[\hat{A}, \hat{B}] = 0$ then the eigenfunctions of $\hat{A}$ and $\hat{B}$ are equal. [2]

   (i) In Dirac notation $|n\rangle$ corresponds to a quantum state of the system with wave function Ψ_n. $\langle n|\hat{Q}|m\rangle = \int \Psi_n \hat{Q} \Psi_m \mathrm{d}x$. [2]

   (j) In matrix notation we represent wave functions by a vector of coefficients. We represent operators by matrices. [2]

2. **Wavefunctions [20]**

   Consider a particle in one dimension whose wave function is $\psi(x) = Nx \exp(-\alpha x^2/2)$.

   (a) Normalization gives

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x \\ &= N^2 \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2) \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= N^2 \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} \\
\Rightarrow N &= \left(\frac{4\alpha^3}{\pi}\right)^{1/4}
\end{aligned}
$$

(b) Expectation value of the position is

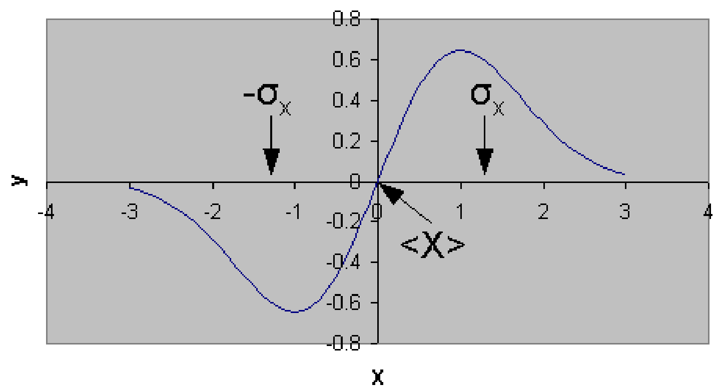
$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x|\psi(x)|^2 \mathrm{d}x \\
&= N^2 \int_{-\infty}^{\infty} x^3 \exp(-\alpha x^2)\mathrm{d}x \\
&= 0
\end{aligned}
$$

where the last line follows from the fact that the integrand is an odd function of x. The probability of finding the particle at 0 is $|\psi(0)|^2 \Delta x = 0$.

(c) The mean uncertainty in the position of the particle is

$$
\begin{aligned}
\sigma_x &= \sqrt{\int_{-\infty}^{\infty} (x - \langle x \rangle)^2 |\psi(x)|^2 \mathrm{d}x} \\
&= \sqrt{\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \mathrm{d}x} \\
&= \sqrt{\int_{-\infty}^{\infty} N^2 x^4 \exp(-\alpha x^2)\mathrm{d}x} \\
&= \left(\frac{4\alpha^3}{\pi}\right)^{1/4} \left(\frac{9\pi}{16\alpha^5}\right)^{1/4} \\
&= \sqrt{\frac{3}{2\alpha}}
\end{aligned}
$$

(d) The graph:



3. **Dirac notation [20]**

(a) $\langle n|n\rangle = 1$ and $\langle n|m\rangle = 0$ for $n \neq m$

(b) $\langle n|\hat{H}|n\rangle = \langle n|\epsilon_n|n\rangle = \epsilon_n$ and $\langle n|\hat{H}|m\rangle = \langle n|\epsilon_m|m\rangle = 0$ for ($n \neq m$)

(c) $\langle\psi|\hat{H}|\psi\rangle = \sum_{nm} C_n^* C_m \langle n|\hat{H}|m\rangle = \sum_n |C_n|^2 \epsilon_n$

(d) The variance of the measured energy is

$$\begin{aligned}
\sigma_H^2 &= \langle\psi|(\hat{H} - \langle H\rangle)^2|\psi\rangle \\
&= \sum_{nm} C_n^* C_m \langle n|(\hat{H} - \langle H\rangle)^2|m\rangle \\
&= \sum_n |C_n|^2 \epsilon_n^2 - \left(\sum_n |C_n|^2 \epsilon_n\right)^2
\end{aligned}$$

(e) We have$\hat{H} = \sum_{nm} a_{nm}|n\rangle\langle m|$ and $\hat{H}|n\rangle = \epsilon_n|n\rangle$. Thus

$$\begin{aligned}
\hat{H}|p\rangle &= \sum_{nm} a_{nm}|n\rangle\langle m|p\rangle \\
&= \sum_n a_{np}|n\rangle \\
\Rightarrow \langle q|\hat{H}|p\rangle &= \sum_n a_{np}\langle q|n\rangle \\
&= a_{qp} \\
\Rightarrow a_{qp} &= \langle q|\epsilon_p|p\rangle \\
&= \epsilon_p \delta_{qp}
\end{aligned}$$

(f) We have $\hat{H}|a\rangle = |A\rangle$ and $\hat{H}|b\rangle = |B\rangle$. Thus $\langle a|\hat{H}^\dagger = \langle A|$ and $\langle b|\hat{H}^\dagger = \langle B|$. If $\hat{H}$ is hermitian then $\hat{H} = \hat{H}^\dagger$ and hence $\langle A|b\rangle = \langle a|\hat{H}^\dagger|b\rangle = \langle a|\hat{H}|b\rangle = \langle a|B\rangle$. If $|a\rangle$ and $|b\rangle$ are eigenstates of the Hamiltonian then $\langle A|b\rangle = \epsilon_a^*\langle a|b\rangle = \langle a|B\rangle = \epsilon_b\langle a|b\rangle$ and hence $0 = (\epsilon_a^* - \epsilon_b)\langle a|b\rangle$ and $0 = (\epsilon_a^* - \epsilon_a)$ since $\langle a|a\rangle > 0$. Thus the eigenvalues are real.