Physics 3C26 - Solutions to Problem Set 2

January 4, 2006

1. **Quick questions [20]**

   (a) $E_n = (n + \frac{1}{2})\hbar\omega$ [2]

   (b) The amplitude. [2]

   (c) $\hat{a}_{\pm} = \frac{1}{\sqrt{2m\hbar\omega}} \left(\hat{p} \pm \mathrm{i}m\omega\hat{x}\right)$ [2]

   (d) $\hat{a}_{+}|n\rangle = \mathrm{i}\sqrt{n+1}|n+1\rangle$ [2]

   (e) $L_x = yp_z - zp_y$ (classical), $\hat{L}_x = -\mathrm{i}\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$ (quantum) [2]

   (f) $[\hat{L}^2, \hat{L}_x] = 0$. The eigenvalues of $\hat{L}^2$are $l(l+1)\hbar^2$. [2]

   (g) $\hat{L}_{\pm} = \hat{L}_x \pm \mathrm{i}\hat{L}_y$ [2]

   (h) The largest and smallest values of m are l and $-l$ respectively. [2]

   (i) The allowed eigenvalues for the electron spin operators $\hat{S}^2$ and $\hat{S}_z$ are $\frac{3}{4}\hbar^2$ and $\pm\frac{1}{2}\hbar$ respectively. [2]

   (j) The commutation relations ($\hat{J} \times \hat{J} = \mathrm{i}\hbar\hat{J}$) [2]

2. **Harmonic Oscillator [20]**

   (a) Starting from the definitions of the raising and lowering operators we get

$$
\begin{aligned}
\hat{a}_{\pm} &= \frac{1}{\sqrt{2m\hbar\omega}} \left(\hat{p} \pm \mathrm{i}m\omega\hat{x}\right) \\
\Rightarrow [\hat{a}_{-}, \hat{a}_{+}] &= \frac{1}{2m\hbar\omega} \left([\hat{p},\hat{p}] + \mathrm{i}m\omega\,[\hat{p},\hat{x}] - \mathrm{i}m\omega\,[\hat{x},\hat{p}] + m^2\omega^2\,[\hat{x},\hat{x}]\right) \\
&= \frac{1}{2m\hbar\omega} \left(0 + \hbar m\omega + \hbar m\omega + 0\right) = 1
\end{aligned}
$$

   [4]

(b) We start from $\hat{a}_-|n\rangle = c_{n-1}|n-1\rangle$

$$\begin{aligned}
\langle n|\hat{a}_-^\dagger \hat{a}_-|n\rangle &= \langle n|\hat{a}_+\hat{a}_-|n\rangle = \langle n|\hat{N}|n\rangle = n \\
&= |c_{n-1}|^2\langle n-1|n-1\rangle = |c_{n-1}|^2 \\
\Rightarrow c_{n-1} &= \mathrm{e}^{\mathrm{i}\phi}\sqrt{n}
\end{aligned}$$

By convention $c_{n-1} = -\mathrm{i}\sqrt{n}$. [4]

(c) We have $\hat{x} = -\mathrm{i}\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ - \hat{a}_-)$. Thus

$$\begin{aligned}
\hat{x}^2 &= -\frac{\hbar}{2m\omega}\left(\hat{a}_+^2 + \hat{a}_-^2 - \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-\right) \\
&= -\frac{\hbar}{2m\omega}\left(\hat{a}_+^2 + \hat{a}_-^2 - 2\hat{a}_+\hat{a}_- - 1\right) \\
\langle n|\hat{a}_-^2|n\rangle &= \langle n|\sqrt{n(n-1)}|n-2\rangle = 0 \\
\langle n|\hat{a}_+^2|n\rangle &= \langle n|\sqrt{(n+1)(n+2)}|n+2\rangle = 0 \\
\langle n|2\hat{a}_+\hat{a}_- + 1|n\rangle &= 2n+1 \\
\Rightarrow \langle n|\frac{1}{2}m\omega^2\hat{x}^2|n\rangle &= \frac{1}{2}(n+\frac{1}{2})\hbar\omega
\end{aligned}$$

[4]

(d) Similarly $\hat{p} = \sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}_+ + \hat{a}_-)$. Thus

$$\begin{aligned}
\hat{p}^2 &= \frac{1}{2}m\hbar\omega\left(\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_-\right) \\
\Rightarrow \langle n|\frac{1}{2m}\hat{p}^2|n\rangle &= \frac{1}{2}(n+\frac{1}{2})\hbar\omega
\end{aligned}$$

[4]

(e) $\langle n|\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2|n\rangle = (n+\frac{1}{2})\hbar\omega$. This follows from the fact that $(n+\frac{1}{2})\hbar\omega$ is just the energy and $\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$ is the Hamiltonian. [4]

3. **Angular Momentum [20]**

(a) We have $\hat{L}_\pm = \hat{L}_x \pm \mathrm{i}\hat{L}_y$. Thus

$$\begin{aligned}
\hat{L}_\pm &= -\mathrm{i}\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \pm \hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\
&= \hbar\left((\mathrm{i}\sin\phi \pm \cos\phi)\,\frac{\partial}{\partial\theta} + \cot\theta\,(\mp\sin\phi + \mathrm{i}\cos\phi)\,\frac{\partial}{\partial\phi}\right) \\
&= \hbar\mathrm{e}^{\pm\mathrm{i}\phi}\left(\mathrm{i}\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right)
\end{aligned}$$

[5]

(b) $Y_{01} \propto \hat{L}_+ Y_{00} = \hbar e^{i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right)\frac{1}{\sqrt{4\pi}} = 0$
$Y_{0-1} \propto \hat{L}_- Y_{00} = \hbar e^{-i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right)\frac{1}{\sqrt{4\pi}} = 0$
This is what we would expect since the only allowed value of m when $l = 0$ is 0. [5]

(c) Using $\hat{L}_+ = \hbar\sqrt{(l-m)(l+m+1)}|lm{+}1\rangle$ we get $\hat{L}_+ Y_{10} = \hbar\sqrt{2}Y_{11}$. But

$$\begin{aligned}
\hat{L}_+ Y_{10} &= \hbar e^{i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right)\sqrt{\frac{3}{4\pi}}\cos\theta \\
&= -\hbar e^{i\phi}\sqrt{\frac{3}{4\pi}}\sin\theta \\
\Rightarrow Y_{11} &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}
\end{aligned}$$

[5]

(d) Using $\hat{L}_- = \hbar\sqrt{(l+m)(l-m+1)}|lm{-}1\rangle$ we get $\hat{L}_+ Y_{10} = \hbar\sqrt{2}Y_{11}$. But

$$\begin{aligned}
\hat{L}_- Y_{10} &= \hbar e^{-i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right)\sqrt{\frac{3}{4\pi}}\cos\theta \\
&= \hbar e^{-i\phi}\sqrt{\frac{3}{4\pi}}\sin\theta \\
\Rightarrow Y_{11} &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}
\end{aligned}$$

[5]